

## Chapter 482

# Mixed Integer Programming

## Introduction

Linear programming maximizes (or minimizes) a linear objective function subject to one or more constraints. Mixed integer programming adds one additional condition that at least one of the variables can only take on integer values. The technique finds broad use in operations research.

The mathematical representation of the mixed integer programming (MIP) problem is

Maximize (or minimize)

$$z = \mathbf{C}\mathbf{X}$$

subject to

$$\mathbf{A}\mathbf{X} \leq \mathbf{b}, \mathbf{X} \geq \mathbf{0}, \text{ some } x_i \text{ are restricted to integer values.}$$

where

$$\mathbf{X} = (x_1, x_2, \dots, x_n)'$$

$$\mathbf{C} = (c_1, c_2, \dots, c_n)$$

$$\mathbf{b} = (b_1, b_2, \dots, b_m)'$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

The  $x_i$ 's are the *decision variables* (the unknowns), the first equation is called the *objective function* and the  $m$  inequalities (and equalities) are called *constraints*. The constraint bounds, the  $b_i$ 's, are often called *right-hand sides* (RHS).

NCSS solves a particular MIP problem using the branch and bound algorithm available in the *Extreme Optimization* mathematical subroutine package.

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**Example**

We will solve the following problem using NCSS:

Maximize

$$z = x_1 + x_2 + 2x_3 - 2x_4$$

subject to

$$x_1 + 2x_3 \leq 700$$

$$2x_2 - 8x_3 \leq 0$$

$$x_2 - 2x_3 + x_4 \geq 1$$

$$x_1 + x_2 + x_3 + x_4 = 10$$

$$0 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

$$0 \leq x_3 \leq 10$$

$$0 \leq x_4 \leq 10$$

All variables are integers.

The solution (see Example 1 below) is  $x_1 = 3$ ,  $x_2 = 4$ ,  $x_3 = 2$ , and  $x_4 = 1$  which results in  $z = 9.0$ . Note that this is quite different from the LP solution of  $x_1 = 9$ ,  $x_2 = 0.8$ ,  $x_3 = 0$ , and  $x_4 = 0.2$  which results in  $z = 9.4$ .

**Data Structure**

This technique requires a special data format which will be discussed under the *Specifications* tab. Here is the way the above example would be entered. It is stored in the dataset *LP 1*.

**LP 1 dataset**

Type	Logic	RHS	X1	X2	X3	X4
O			1	1	2	-2
C	<	700	1		2	
C	<	0		2		-8
C	>	1		1	-2	1
C	=	10	1	1	1	1
L			0	0	0	0
U			10	10	10	10

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## Procedure Options

This section describes the options available in this procedure.

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### Specifications Tab

Set the specifications for the analysis.

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#### Optimum Type

##### Type of Optimum

Specify whether to find the minimum or the maximum of the objective function in the constrained region.

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#### Coefficients of the Objective Function, Constraints, and Bounds

##### Row Type Column

The constrained minimization problem is specified on the spreadsheet. In this column, you indicate the type of information (objective function, constraint, upper bound, or lower bound) that is on each row by entering one of the four letters: O, C, U, or L. Note that the order of the rows does not matter.

The possible cell entries in this column are

- **O**  
This row contains the objective function defined by the coefficients (costs).
- **C**  
This row contains a constraint defined by logic (<, >, or =), the RHS, and the coefficients.
- **L**  
This row contains the optional lower bound for each variable.
- **U**  
This row contains the optional upper bound for each variable.

##### Continuous Variables Columns

Specify the columns containing the coefficients of the variables that are not required to be integers. Use of these variables is optional. Each column corresponds to a variable in the constraints and objective function. Each row represents an individual constraint, the objective function, or upper or lower bounds. The program interprets the rows according to the corresponding values of the Row Type column. The coefficients can be either positive or negative. Note that blanks are treated as zeros.

If you to enter upper bounds or lower bounds, enter them as rows: one for the upper bounds and another for the lower bounds.

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For example, consider a problem to minimize the weighted sum of two variables X and Y subject to several constraints. Here is the generic formulation.

$$\begin{aligned} \text{Minimize:} \quad & X + 2Y \\ \text{s.t.} \quad & \\ & X - Y \leq 3 \\ & X + 2Y \geq 6 \\ & 3X - 4Y = 10 \\ \text{with bounds} \quad & \\ & 4 \leq X \leq 20 \\ & 2 \leq Y \leq 22 \end{aligned}$$

This is entered in five columns on the spreadsheet as follows.

C1	C2	C3	C4	C5
O			1	2
C	<	3	1	-1
C	>	6	1	2
C	=	10	3	-4
L			4	2
U			20	22

### Integer Variables Columns

Specify the columns containing the coefficients of the variables that are required to be integers. At least one integer or binary variable must be selected. Each column corresponds to a variable in the constraints and objective function. Each row represents an individual constraint, the objective function, or upper or lower bounds. The program interprets the rows according to the corresponding values of the Row Type column. The coefficients can be either positive or negative. Note that blanks are treated as zeros.

If you to enter upper bounds or lower bounds, enter them as rows: one for the upper bounds and another for the lower bounds.

### Binary Variables Columns

Specify the columns containing the coefficients of the variables that are required to be integers limited to two values, 0 or 1. At least one integer or binary variable must be selected. Each column corresponds to a variable in the constraints and objective function. Each row represents an individual constraint, the objective function, or upper or lower bounds. The program interprets the rows according to the corresponding values of the Row Type column. The coefficients can be either positive or negative. Note that blanks are treated as zeros.

If you to enter upper bounds or lower bounds, enter them as rows: one for the upper bounds and another for the lower bounds.

### Labels of Constraints Column

Specify a column containing a label for each constraint (*Row Type* = 'C'). These labels are used to make the output of this procedure easier to interpret.

Labels in non-constraint rows are ignored.

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### Logic and Constraint Bounds (RHS)

#### Logic Column

Specify the column containing the logic values for the constraints. Note that the cells in this column are only used for constraints (rows whose *Row Type* is "C").

#### Possible Logic Values

- <  
For less than or equal. You can also use "LE" or "<=".  
**Example:  $X1 + X2 \leq 6$ .**
- >  
For greater than or equal. You can also use "GE" or ">=".  
**Example:  $X1 + X2 \geq 4$ .**
- =  
For equal. You can also use "EQ".  
**Example:  $X1 + X2 = 3$ .**

#### Constraint Bounds (RHS) Column

Specify the bound (RHS = right-hand-side) of each of the constraints, one per row.

See the Example in the *Variables Columns* help.

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## Reports Tab

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### Select Reports

#### Objective Function and Solution – Values of Constraints

Indicate which reports you want to view.

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### Report Options

#### Variable Names

This option lets you select whether to display only variable names, variable labels, or both.

#### Precision

Specify the precision of numbers in the report. Single precision will display seven-place accuracy, while double precision will display thirteen-place accuracy.

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### Report Options – Decimal Places

#### Input Coefficients – Calculated Values

These options let you designate the number of decimal places to be displayed for each type of variable.

## Example 1 – Mixed Integer Programming

This section presents an example of how to run the data presented in the example given above. The data are contained in the LP 1 database. Here is the specification of the problem.

Maximize

$$z = x_1 + x_2 + 2x_3 - 2x_4$$

subject to

$$x_1 + 2x_3 \leq 700$$

$$2x_2 - 8x_3 \leq 0$$

$$x_2 - 2x_3 + x_4 \geq 1$$

$$x_1 + x_2 + x_3 + x_4 = 10$$

$$0 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

$$0 \leq x_3 \leq 10$$

$$0 \leq x_4 \leq 10$$

All variables are integers.

You may follow along here by making the appropriate entries or load the completed template **Example 1** by clicking on Open Example Template from the File menu of the Mixed Integer Programming window.

### 1 Open the LP 1 dataset.

- From the **File** menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **LP 1.NCSS**.
- Click **Open**.

### 2 Open the Mixed Integer Programming window.

- Using the Analysis menu or the Procedure Navigator, find and select the **Mixed Integer Programming** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 3 Specify the problem.

- On the Mixed Integer Programming window, select the **Specifications tab**.
- Set **Type of Optimum** to **Maximum**.
- Double-click in the **Row Type Column** text box. This will bring up the column selection window.
- Select variables **Type** from the list of columns and then click **Ok**. “Type” will appear in this box.
- Double-click in the **Integer Variables Columns** text box. This will bring up the column selection window.
- Select columns **X1-X4** from the list of columns and then click **Ok**. “X1-X4” will appear in this box.
- Double-click in the **Labels of Constraints Column** text box. This will bring up the column selection window.
- Select **CLabel** from the list of columns and then click **Ok**. “CLabel” will appear in this box.
- Double-click in the **Logic Column** text box. This will bring up the column selection window.
- Select **Logic** from the list of columns and then click **Ok**. “Logic” will appear in this box.
- Double-click in the **Constraint Bounds (RHS) Column** text box. This will bring up the variable selection window.
- Select column **RHS** from the list of columns and then click **Ok**. “RHS” will appear in this box.

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**4 Run the procedure.**

- From the **Run** menu, select **Run Procedure**. Alternatively, just click the green Run button.

**Objective Function and Solution for Maximum**

## Objective Function and Solution for Maximum

Variable	Type	Objective Function Coefficient	Value at Maximum
X1	Integer	1.0	3.000
X2	Integer	1.0	4.000
X3	Integer	2.0	2.000
X4	Integer	-2.0	1.000

Maximum of Objective Function 9.000

Solution Status: The optimization model is optimal.

This report lists the linear portion of the objective function coefficients and the values of the variables at the maximum (that is, the solution). It also shows the value of the objective function at the solution as well as the status of the algorithm when it terminated.

**Constraints**

## Constraints

Label, Logic	X1	X2	X3	X4	RHS
Con1, ≤	1.0	0.0	2.0	0.0	700.0
Con2, ≤	0.0	2.0	0.0	-8.0	0.0
Con3, ≥	0.0	1.0	-2.0	1.0	1.0
Con4, =	1.0	1.0	1.0	1.0	10.0

This report presents the coefficients of the constraints as they were input.

**Values of Constraints at Solution for Maximum**

## Values of Constraints at Solution for Maximum

Row, Logic	RHS	RHS at Solution
2, ≤	700.0	7.000
3, ≤	0.0	0.000
4, ≥	1.0	1.000
5, =	10.0	10.000

This report presents the right hand side of each constraint along with its value at the optimal values of the variables.