

## Chapter 304

# Multiple Regression – Basic

## Introduction

*Multiple Regression Analysis* refers to a set of techniques for studying the straight-line relationships among two or more variables. Multiple regression estimates the  $\beta$ 's in the equation

$$y_j = \beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j} + \cdots + \beta_p x_{pj} + \varepsilon_j$$

The  $X$ 's are the *independent variables* (IV's).  $Y$  is the *dependent variable*. The subscript  $j$  represents the observation (row) number. The  $\beta$ 's are the unknown *regression coefficients*. Their estimates are represented by  $b$ 's. Each  $\beta$  represents the original unknown (population) parameter, while  $b$  is an estimate of this  $\beta$ . The  $\varepsilon_j$  is the error (residual) of observation  $j$ .

Although the regression problem may be solved by a number of techniques, the most-used method is least squares. In least squares regression analysis, the  $b$ 's are selected so as to minimize the sum of the squared residuals. This set of  $b$ 's is not necessarily the set you want, since they may be distorted by *outliers*--points that are not representative of the data. Robust regression, an alternative to least squares, seeks to reduce the influence of outliers.

Multiple regression analysis studies the relationship between a *dependent* (response) *variable* and  $p$  *independent variables* (*predictors*, *regressors*, *IV's*). The sample multiple regression equation is

$$\hat{y}_j = b_0 + b_1 x_{1j} + b_2 x_{2j} + \cdots + b_p x_{pj}$$

If  $p = 1$ , the model is called *simple linear regression*.

The intercept,  $b_0$ , is the point at which the regression plane intersects the  $Y$  axis. The  $b_i$  are the slopes of the regression plane in the direction of  $x_i$ . These coefficients are called the *partial-regression coefficients*. Each partial regression coefficient represents the net effect the  $i^{\text{th}}$  variable has on the dependent variable, holding the remaining  $X$ 's in the equation constant.

A large part of a regression analysis consists of analyzing the sample *residuals*,  $e_j$ , defined as

$$e_j = y_j - \hat{y}_j$$

Once the  $\beta$ 's have been estimated, various indices are studied to determine the reliability of these estimates. One of the most popular of these reliability indices is the correlation coefficient. The correlation coefficient, or simply the correlation, is an index that ranges from -1 to 1. When the value is near zero, there is no linear relationship. As the correlation gets closer to plus or minus one, the relationship is stronger. A value of one (or negative one) indicates a perfect linear relationship between two variables.

The regression equation is only capable of measuring linear, or straight-line, relationships. If the data form a circle, for example, regression analysis would not detect a relationship. For this reason, it is always advisable to plot each independent variable with the dependent variable, watching for curves, outlying points, changes in the amount of variability, and various other anomalies that may occur.

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If the data are a random sample from a larger population and the  $\varepsilon_j$  are independent and normally distributed, a set of statistical tests may be applied to the  $b$ 's and the correlation coefficient. These  $t$ -tests and  $F$ -tests are valid only if the above assumptions are met.

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## Regression Models

In order to make good use of multiple regression, you must have a basic understanding of the regression model. The basic regression model is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

This expression represents the relationship between the dependent variable (DV) and the independent variables (IV's) as a weighted average in which the regression coefficients ( $\beta$ 's) are the weights. Unlike the usual weights in a weighted average, it is possible for the regression coefficients to be negative.

A fundamental assumption in this model is that the effect of each IV is additive. Now, no one really believes that the true relationship is actually additive. Rather, they believe that this model is a reasonable first-approximation to the true model. To add validity to this approximation, you might consider this additive model to be a Taylor-series expansion of the true model. However, this appeal to the Taylor-series expansion usually ignores the 'local-neighborhood' assumption.

Another assumption is that the relationship of the DV with each IV is linear (straight-line). Here again, no one really believes that the relationship is a straight-line. However, this is a reasonable first approximation.

In order to obtain better approximations, methods have been developed to allow regression models to approximate curvilinear relationships as well as non-additivity. Although nonlinear regression models can be used in these situations, they add a higher level of complexity to the modeling process. An experienced user of multiple regression knows how to include curvilinear components in a regression model when it is needed.

Another issue is how to add categorical variables into the model. Unlike regular numeric variables, categorical variables may be alphabetic. Examples of categorical variables are gender, producer, and location. In order to effectively use multiple regression, you must know how to include categorical IV's in your regression model.

This section shows how NCSS may be used to specify and estimate advanced regression models that include curvilinearity, interaction, and categorical variables.

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## Representing a Curvilinear Relationship

A curvilinear relationship between a DV and one or more IV's is often modeled by adding new IV's which are created from the original IV by squaring, and occasionally cubing, them. For example, the regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

might be expanded to

$$\begin{aligned} Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_2^2 + \beta_5 X_1 X_2 \\ &= \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 \end{aligned}$$

Note that this model is still additive in terms of the new IV's.

One way to adopt such a new model is to create the new IV's using the transformations of existing variables.

## Representing Categorical Variables

Categorical variables take on only a few unique values. For example, suppose a therapy variable has three possible values: A, B, and C. One question is how to include this variable in the regression model. At first glance, we can convert the letters to numbers by recoding A to 1, B to 2, and C to 3. Now we have numbers. Unfortunately, we will obtain completely different results if we recode A to 2, B to 3, and C to 1. Thus, a direct recode of letters to numbers will not work.

To convert a categorical variable to a form usable in regression analysis, we have to create a new set of numeric variables. If a categorical variable has  $k$  values,  $k - 1$  new variables must be generated.

There are many ways in which these new variables may be generated. You can use the Contrasts data tool in NCSS (Data Window Menu: Data > Create Contrast Variables) to automatically create many types of contrasts and binary indicator variables. We will present a few examples here.

### Indicator Variables

Indicator (dummy or binary) variables are a popular type of generated variables. They are created as follows. A *reference value* is selected. Usually, the most common value is selected as the reference value. Next, a variable is generated for each of the values other than the reference value. For example, suppose that C is selected as the reference value. An indicator variable is generated for each of the remaining values: A and B. The value of the indicator variable is one if the value of the original variable is equal to the value of interest, or zero otherwise. Here is how the original variable T and the two new indicator variables TA and TB look in a short example.

<u>T</u>	<u>TA</u>	<u>TB</u>
A	1	0
A	1	0
B	0	1
B	0	1
C	0	0
C	0	0

The generated IV's, TA and TB, would be used in the regression model.

### Contrast Variables

Contrast variables are another popular type of generated variables. Several types of contrast variables can be generated. We will present a few here. One method is to contrast each value with the reference value. The value of interest receives a one. The reference value receives a negative one. All other values receive a zero.

Continuing with our example, one set of contrast variables is

<u>T</u>	<u>CA</u>	<u>CB</u>
A	1	0
A	1	0
B	0	1
B	0	1
C	-1	-1
C	-1	-1

The generated IV's, CA and CB, would be used in the regression model.

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Another set of contrast variables that is commonly used is to compare each value with those remaining. For this example, we will suppose that  $T$  takes on four values: A, B, C, and D. The generate variables are

<u>T</u>	<u>C1</u>	<u>C2</u>	<u>C3</u>
A	-3	0	0
A	-3	0	0
B	1	-2	0
B	1	-2	0
C	1	1	-1
C	1	1	-1
D	1	1	1
D	1	1	1

Many other methods have been developed to provide meaningful numeric variables that represent categorical variable. We have presented these because they may be generated automatically by NCSS.

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## Representing Interactions of Numeric Variables

The interaction between two variables is represented in the regression model by creating a new variable that is the product of the variables that are interacting. Suppose you have two variables  $X1$  and  $X2$  for which an interaction term is necessary. A new variable is generated by multiplying the values of  $X1$  and  $X2$  together.

<u>X1</u>	<u>X2</u>	<u>Int</u>
1	1	1
2	1	2
3	2	6
2	2	4
0	4	0
5	-2	-10

The new variable, *Int*, is added to the regression equation and treated like any other variable during the analysis. With *Int* in the regression model, the interaction between  $X1$  and  $X2$  may be investigated.

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## Representing Interactions of Numeric and Categorical Variables

When the interaction between a numeric IV and a categorical IV is to be included in the model, all proceeds as above, except that an interaction variable must be generated for each categorical variable.

In the following example, the interaction between the categorical variable  $T$  and the numeric variable  $X$  is created.

<u>T</u>	<u>CA</u>	<u>CB</u>	<u>X</u>	<u>XCA</u>	<u>XCB</u>
A	1	0	1.2	1.2	0
A	1	0	1.4	1.4	0
B	0	1	2.3	0	2.3
B	0	1	4.7	0	4.7
C	-1	-1	3.5	-3.5	-3.5
C	-1	-1	1.8	-1.8	-1.8

When the variables  $XCA$  and  $XCB$  are added to the regression model, they will account for the interaction between  $T$  and  $X$ .

## Representing Interactions Two or More Categorical Variables

When the interaction between two categorical variables is included in the model, an interaction variable must be generated for each combination of the variables generated for each categorical variable.

In the following example, the interaction between the categorical variables *T* and *S* are generated. Try to determine the reference value used for variable *S*.

<u>T</u>	<u>CA</u>	<u>CB</u>	<u>S</u>	<u>S1</u>	<u>S2</u>	<u>CAS1</u>	<u>CAS2</u>	<u>CBS1</u>	<u>CBS2</u>
A	1	0	D	1	0	1	0	0	0
A	1	0	E	0	1	0	1	0	0
B	0	1	F	0	0	0	0	0	0
B	0	1	D	1	0	0	0	1	0
C	-1	-1	E	0	1	0	-1	0	-1
C	-1	-1	F	0	0	0	0	0	0

When the variables, *CAS1*, *CAS2*, *CBS1*, and *CBS2* are added to the regression model, they will account for the interaction between *T* and *S*.

## Possible Uses of Regression Analysis

Montgomery (1982) outlines the following five purposes for running a regression analysis.

### Description

The analyst is seeking to find an equation that describes or summarizes the relationships in a set of data. This purpose makes the fewest assumptions.

### Coefficient Estimation

This is a popular reason for doing regression analysis. The analyst may have a theoretical relationship in mind, and the regression analysis will confirm this theory. Most likely, there is specific interest in the magnitudes and signs of the coefficients. Frequently, this purpose for regression overlaps with others.

### Prediction

The prime concern here is to predict some response variable, such as sales, delivery time, efficiency, occupancy rate in a hospital, reaction yield in some chemical process, or strength of some metal. These predictions may be very crucial in planning, monitoring, or evaluating some process or system. There are many assumptions and qualifications that must be made in this case. For instance, you must not extrapolate beyond the range of the data. Also, interval estimates require special, so-called normality, assumptions to hold.

### Control

Regression models may be used for monitoring and controlling a system. For example, you might want to calibrate a measurement system or keep a response variable within certain guidelines. When a regression model is used for control purposes, the independent variables must be related to the dependent in a causal way. Furthermore, this functional relationship must continue over time. If it does not, continual modification of the model must occur.

### Variable Selection or Screening

In this case, a search is conducted for those independent variables that explain a significant amount of the variation in the dependent variable. In most applications, this is not a one-time process but a continual model-

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building process. This purpose is manifested in other ways, such as using historical data to identify factors for future experimentation.

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### Assumptions

The following assumptions must be considered when using multiple regression analysis.

#### Linearity

Multiple regression models the linear (straight-line) relationship between  $Y$  and the  $X$ 's. Any curvilinear relationship is ignored. This is most easily evaluated by scatter plots early on in your analysis. Nonlinear patterns can show up in residual plots.

#### Constant Variance

The variance of the  $\varepsilon$ 's is constant for all values of the  $X$ 's. This can be detected by residual plots of  $e_j$  versus  $\hat{y}_j$  or the  $X$ 's. If these residual plots show a rectangular shape, we can assume constant variance. On the other hand, if a residual plot shows an increasing or decreasing wedge or bowtie shape, non-constant variance exists and must be corrected.

#### Special Causes

We assume that all special causes, outliers due to one-time situations, have been removed from the data. If not, they may cause non-constant variance, non-normality, or other problems with the regression model.

#### Normality

We assume the  $\varepsilon$ 's are normally distributed when hypothesis tests and confidence limits are to be used.

#### Independence

The  $\varepsilon$ 's are assumed to be uncorrelated with one another, which implies that the  $Y$ 's are also uncorrelated. This assumption can be violated in two ways: model misspecification or time-sequenced data.

1. *Model misspecification.* If an important independent variable is omitted or if an incorrect functional form is used, the residuals may not be independent. The solution to this dilemma is to find the proper functional form or to include the proper independent variables.
2. *Time-sequenced data.* Whenever regression analysis is performed on data taken over time (frequently called time series data), the residuals are often correlated. This correlation among residuals is called serial correlation or autocorrelation. Positive autocorrelation means that the residual in time period  $j$  tends to have the same sign as the residual in time period  $(j-k)$ , where  $k$  is the lag in time periods. On the other hand, negative autocorrelation means that the residual in time period  $j$  tends to have the opposite sign as the residual in time period  $(j-k)$ .

The presence of autocorrelation among the residuals has several negative impacts:

1. The regression coefficients are unbiased but no longer efficient, i.e., minimum variance estimates.
2. With positive serial correlation, the mean square error may be seriously underestimated. The impact of this is that the standard errors are underestimated, the partial t-tests are inflated (show significance when there is none), and the confidence intervals are shorter than they should be.
3. Any hypothesis tests or confidence limits that required the use of the t or F distribution would be invalid.

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You could try to identify these serial correlation patterns informally, with the residual plots versus time. A better analytical way would be to compute the serial or autocorrelation coefficient for different time lags and compare it to a critical value.

### Multicollinearity

Collinearity, or multicollinearity, is the existence of near-linear relationships among the set of independent variables. The presence of multicollinearity causes all kinds of problems with regression analysis, so you could say that we assume the data do not exhibit it.

#### Effects of Multicollinearity

Multicollinearity can create inaccurate estimates of the regression coefficients, inflate the standard errors of the regression coefficients, deflate the partial  $t$ -tests for the regression coefficients, give false nonsignificant  $p$ -values, and degrade the predictability of the model.

#### Sources of Multicollinearity

To deal with collinearity, you must be able to identify its source. The source of the collinearity impacts the analysis, the corrections, and the interpretation of the linear model. There are five sources (see Montgomery [1982] for details):

1. *Data collection.* In this case, the data has been collected from a narrow subspace of the independent variables. The collinearity has been created by the sampling methodology. Obtaining more data on an expanded range would cure this collinearity problem.
2. *Physical constraints* of the linear model or population. This source of collinearity will exist no matter what sampling technique is used. Many manufacturing or service processes have constraints on independent variables (as to their range), either physically, politically, or legally, which will create collinearity.
3. *Over-defined model.* Here, there are more variables than observations. This situation should be avoided.
4. *Model choice or specification.* This source of collinearity comes from using independent variables that are higher powers or interactions of an original set of variables. It should be noted that if sampling subspace of  $X_j$  is narrow, then any combination of variables with  $x_j$  will increase the collinearity problem even further.
5. *Outliers.* Extreme values or outliers in the  $X$ -space can cause collinearity as well as hide it.

#### Detection of Collinearity

The following steps for detecting collinearity proceed from simple to complex.

1. Begin by studying pairwise scatter plots of pairs of independent variables, looking for near-perfect relationships. Also glance at the correlation matrix for high correlations. Unfortunately, multicollinearity does not always show up when considering the variables two at a time.
2. Next, consider the variance inflation factors (*VIF*). Large *VIF*'s flag collinear variables.
3. Finally, focus on small eigenvalues of the correlation matrix of the independent variables. An eigenvalue of zero or close to zero indicates that an exact linear dependence exists. Instead of looking at the numerical size of the eigenvalue, use the condition number. Large condition numbers indicate collinearity.

#### Correction of Collinearity

Depending on what the source of collinearity is, the solutions will vary. If the collinearity has been created by the data collection, then collect additional data over a wider  $X$ -subspace. If the choice of the linear model has accentuated the collinearity, simplify the model by variable selection techniques. If an observation or two has induced the collinearity, remove those observations and proceed accordingly. Above all, use care in selecting the variables at the outset.

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### Centering and Scaling Issues in Collinearity

When the variables in regression are centered (by subtracting their mean) and scaled (by dividing by their standard deviation), the resulting  $X'X$  matrix is in correlation form. The centering of each independent variable has removed the constant term from the collinearity diagnostics. Scaling and centering permit the computation of the collinearity diagnostics on standardized variables. On the other hand, there are many regression applications where the intercept is a vital part of the linear model. The collinearity diagnostics on the uncentered data may provide a more realistic picture of the collinearity structure in these cases.

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## Multiple Regression Checklist

This checklist, prepared by a professional statistician, is a flowchart of the steps you should complete to conduct a valid multiple regression analysis. Several of these steps should be performed prior to this phase of the regression analysis, but they are briefly listed here again as a reminder. Some of the items may require the use of the more advanced Multiple Regression procedure in NCSS. You should complete these tasks in order.

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### Step 1 – Data Preparation

Scan your data for anomalies, keypunch errors, typos, and so on. You should have a minimum of five observations for each variable in the analysis, including the dependent variable. This discussion assumes that the pattern of missing values is random. All data preparation should be done prior to the use of one of the variable selection strategies.

Special attention must be paid to categorical IV's to make certain that you have chosen a reasonable method of converting them to numeric values.

Also, you must decide how complicated of a model to use. Do you want to include powers of variables and interactions between terms?

One of the best ways to accomplish this data preparation is to run your data through the Data Screening procedure, since it provides reports about missing value patterns, discrete and continuous variables, and so on.

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### Step 2 – Variable Selection

Variable selection seeks to reduce the number of IV's to a manageable few. There are several variable selection methods in regression: Subset Selection, Stepwise Regression, All Possible Regressions, or Multivariate Variable Selection. Each of these variable selection methods has advantages and disadvantages. We suggest that you begin with the Subset Select procedure since it allows you to look at interactions, powers, and categorical variables.

It is extremely important that you complete Step 1 before beginning this step, since variable selection can be greatly distorted by outliers. Every effort should be taken to find outliers before beginning this step.

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### Step 3 – Setup and Run the Regression

#### Introduction

Now comes the fun part: running the program. NCSS is designed to be simple to operate, but it can still seem complicated. When you go to run a procedure such as this for the first time, take a few minutes to read through the chapter again and familiarize yourself with the issues involved.



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### Enter Variables

The NCSS panels are set with ready-to-run defaults, but you have to select the appropriate variables (columns of data). There should be only one dependent variable and one or more independent variables enumerated. In addition, if a weight variable is available from a previous analysis, it needs to be specified.

### Choose Report Options

In multiple linear regression, there is a wide assortment of report options available. As a minimum, you are interested in the coefficients for the regression equation, the analysis of variance report, normality testing, serial correlation (for time-sequenced data), regression diagnostics (looking for outliers), and multicollinearity insights.

### Specify Alpha

Most beginners at statistics forget this important step and let the alpha value default to the standard 0.05. You should make a conscious decision as to what value of alpha is appropriate for your study. The 0.05 default came about during the dark ages when people had to rely on printed probability tables and there were only two values available: 0.05 or 0.01. Now you can set the value to whatever is appropriate.

### Select All Plots

As a rule, select all residual plots. They add a great deal to your analysis of the data.

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## Step 4 – Check Model Adequacy

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Once the regression output is displayed, you will be tempted to go directly to the probability of the  $F$ -test from the regression analysis of variance table to see if you have a significant result. However, it is very important that you proceed through the output in an orderly fashion. The main conditions to check for relate to linearity, normality, constant variance, independence, outliers, multicollinearity, and predictability. Return to the statistical sections and plot descriptions for more detailed discussions.

### Check 1. Linearity

- Look at the Residual vs. Predicted plot. A curving pattern here indicates nonlinearity.
- Look at the Residual vs. Predictor plots. A curving pattern here indicates nonlinearity.
- Look at the  $Y$  versus  $X$  plots. For simple linear regression, a linear relationship between  $Y$  and  $X$  in a scatter plot indicates that the linearity assumption is appropriate. The same holds if the dependent variable is plotted against each independent variable in a scatter plot.
- If linearity does not exist, take the appropriate action and return to Step 2. Appropriate action might be to add power terms (such as  $\text{Log}(X)$ ,  $X$  squared, or  $X$  cubed) or to use an appropriate nonlinear model.

### Check 2. Normality

- Look at the *Normal Probability Plot*. If all of the residuals fall within the confidence bands for the *Normal Probability Plot*, the normality assumption is likely met. One or two residuals outside the confidence bands may be an indicator of outliers, not non-normality.
- Look at the *Normal Assumptions Section*. The formal normal goodness of fit tests are given in the *Normal Assumptions Section*. If the decision is accepted for the *Normality (Omnibus)* test, there is no evidence that the residuals are not normal.

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- If normality does not exist, take the appropriate action and return to Step 2. Appropriate action includes removing outliers and/or using the logarithm of the dependent variable.

### Check 3. Non-constant Variance

- Look at the Residual vs. Predicted plot. If the Residual vs. Predicted plot shows a rectangular shape instead of an increasing or decreasing wedge or a bowtie, the variance is constant.
- Look at the Residual vs. Predictor plots. If the Residual vs. Predictor plots show a rectangular shape, instead of an increasing or decreasing wedge or a bowtie, the variance is constant.
- If non-constant variance does not exist, take the appropriate action and return to Step 2. Appropriate action includes taking the logarithm of the dependent variable or using weighted regression.

### Check 4. Independence or Serial Correlation

- If you have time series data, look at the Serial-Correlations Section. If none of the serial correlations in the Serial-Correlations Section are greater than the critical value that is provided, independence may be assumed.
- Look at the Residual vs. Row plot. A visualization of what the Serial-Correlations Section shows will be exhibited by adjacent residuals being similar (a roller coaster trend) or dissimilar (a quick oscillation).
- If independence does not exist, use a first difference model and return to Step 2. More complicated choices require time series models.

### Check 5. Outliers

- Look at the Regression Diagnostics Section. Any observations with an asterisk by the diagnostics RStudent, Hat Diagonal, DFFITS, or the CovRatio, are potential outliers. Observations with a Cook's  $D$  greater than 1.00 are also potentially influential.
- Look at the Dfbetas Section. Any Dfbetas beyond the cutoff of  $\pm 2/\sqrt{N}$  indicate influential observations.
- Look at the Rstudent vs. Hat Diagonal plot. This plot will flag an observation that may be jointly influential by both diagnostics.
- If outliers do exist in the model, go to robust regression and run one of the options there to confirm these outliers. If the outliers are to be deleted or down weighted, return to Step 2.

### Check 6. Multicollinearity

- Look at the Multicollinearity Section. If any variable has a variance inflation factor greater than 10, collinearity could be a problem.
- Look at the Eigenvalues of Centered Correlations Section. Condition numbers greater than 1000 indicate severe collinearity. Condition numbers between 100 and 1000 imply moderate to strong collinearity.
- Look at the Correlation Matrix Section. Strong pairwise correlation here may give some insight as to the variables causing the collinearity.
- If multicollinearity does exist in the model, it could be due to an outlier (return to Check 5 and then Step 2) or due to strong interdependencies between independent variables. In the latter case, return to Step 2 and try a different variable selection procedure.

### Check 7. Predictability

- Look at the PRESS Section. If the Press  $R^2$  is almost as large as the  $R^2$ , you have done as well as could be expected. It is not unusual in practice for the Press  $R^2$  to be half of the  $R^2$ . If  $R^2$  is 0.50, a Press  $R^2$  of 0.25 would be unacceptable.

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- Look at the Predicted Values with Confidence Limits for Means and Individuals. If the confidence limits are too wide to be practical, you may need to add new variables or reassess the outlier and collinearity possibilities.
- Look at the Residual Report. Any observation that has percent error grossly deviant from the values of most observations is an indication that this observation may be impacting predictability.
- Any changes in the model due to poor predictability require a return to Step 2.

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### Step 5 – Record Your Results

Since multiple regression can be quite involved, it is best make notes of why you did what you did at different steps of the analysis. Jot down what decisions you made and what you have found. Explain what you did, why you did it, what conclusions you reached, which outliers you deleted, areas for further investigation, and so on. Be sure to examine the following sections closely and in the indicated order:

1. Analysis of Variance Section. Check for the overall significance of the model.
2. Regression Equation and Coefficient Sections. Significant individual variables are noted here.

Regression analysis is a complicated statistical tool that frequently demands revisions of the model. Your notes of the analysis process as well as of the interpretation will be worth their weight in gold when you come back to an analysis a few days later!

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## Multiple Regression Technical Details

This section presents the technical details of least squares regression analysis using a mixture of summation and matrix notation.

Define the following vectors and matrices:

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_j \\ \vdots \\ y_N \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ \vdots & \vdots & & \vdots \\ 1 & x_{1j} & \cdots & x_{pj} \\ \vdots & \vdots & & \vdots \\ 1 & x_{1N} & \cdots & x_{pN} \end{bmatrix}, \mathbf{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_N \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{bmatrix}$$

### Least Squares

Using this notation, the least squares estimates are found using the equation.

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \mathbf{Y}$$

The predicted values of the dependent variable are given by

$$\hat{\mathbf{Y}} = \mathbf{b}' \mathbf{X}$$

The residuals are calculated using

$$\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}}$$

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### Estimated Variances

An estimate of the variance of the residuals is computed using

$$s^2 = \frac{\mathbf{e}'\mathbf{e}}{N - p - 1}$$

An estimate of the variance of the regression coefficients is calculated using

$$V \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{pmatrix} = s^2 (\mathbf{X}'\mathbf{X})^{-1}$$

An estimate of the variance of the predicted mean of  $Y$  at a specific value of  $X$ , say  $X_0$ , is given by

$$s_{Y_m|X_0}^2 = s^2 (1, X_0) (\mathbf{X}'\mathbf{X})^{-1} \begin{pmatrix} 1 \\ X_0 \end{pmatrix}$$

An estimate of the variance of the predicted value of  $Y$  for an individual for a specific value of  $X$ , say  $X_0$ , is given by

$$s_{Y_i|X_0}^2 = s^2 + s_{Y_m|X_0}^2$$

### Hypothesis Tests of the Intercept and Slopes

Using these variance estimates and assuming the residuals are normally distributed, hypothesis tests may be constructed using the Student's  $t$  distribution with  $N - p - 1$  degrees of freedom using

$$t_{b_i} = \frac{b_i - B_i}{s_{b_i}}$$

Usually, the hypothesized value of  $B_i$  is zero, but this does not have to be the case.

### Confidence Intervals of the Intercept and Slope

A  $100(1 - \alpha)\%$  confidence interval for the true regression coefficient,  $\beta_i$ , is given by

$$b_i \pm (t_{1-\alpha/2, N-p-1}) s_{b_i}$$

### Confidence Interval of Y for Given X

A  $100(1 - \alpha)\%$  confidence interval for the mean of  $Y$  at a specific value of  $X$ , say  $X_0$ , is given by

$$b'X_0 \pm (t_{1-\alpha/2, N-p-1}) s_{Y_m|X_0}$$

A  $100(1 - \alpha)\%$  prediction interval for the value of  $Y$  for an individual at a specific value of  $X$ , say  $X_0$ , is given by

$$b'X_0 \pm (t_{1-\alpha/2, N-p-1}) s_{Y_i|X_0}$$

## Multiple Regression – Basic

 **$R^2$  (Percent of Variation Explained)**

Several measures of the goodness-of-fit of the regression model to the data have been proposed, but by far the most popular is  $R^2$ .  $R^2$  is the square of the correlation coefficient between  $Y$  and  $\hat{Y}$ . It is the proportion of the variation in  $Y$  that is accounted for by the variation in the independent variables.  $R^2$  varies between zero (no linear relationship) and one (perfect linear relationship).

$R^2$ , officially known as the *coefficient of determination*, is defined as the sum of squares due to the regression divided by the adjusted total sum of squares of  $Y$ . The formula for  $R^2$  is

$$R^2 = 1 - \left( \frac{\mathbf{e}'\mathbf{e}}{\mathbf{Y}'\mathbf{Y} - \frac{(\mathbf{1}'\mathbf{Y})^2}{\mathbf{1}'\mathbf{1}}} \right)$$

$$= \frac{SS_{Model}}{SS_{Total}}$$

$R^2$  is probably the most popular measure of how well a regression model fits the data.  $R^2$  may be defined either as a ratio or a percentage. Since we use the ratio form, its values range from zero to one. A value of  $R^2$  near zero indicates no linear relationship, while a value near one indicates a perfect linear fit. Although popular,  $R^2$  should not be used indiscriminately or interpreted without scatter plot support. Following are some qualifications on its interpretation:

1. *Additional independent variables.* It is possible to increase  $R^2$  by adding more independent variables, but the additional independent variables may actually cause an increase in the mean square error, an unfavorable situation. This usually happens when the sample size is small.
2. *Range of the independent variables.*  $R^2$  is influenced by the range of the independent variables.  $R^2$  increases as the range of the  $X$ 's increases and decreases as the range of the  $X$ 's decreases.
3. *Slope magnitudes.*  $R^2$  does not measure the magnitude of the slopes.
4. *Linearity.*  $R^2$  does not measure the appropriateness of a linear model. It measures the strength of the linear component of the model. Suppose the relationship between  $X$  and  $Y$  was a perfect sphere. Although there is a perfect relationship between the variables, the  $R^2$  value would be zero.
5. *Predictability.* A large  $R^2$  does not necessarily mean high predictability, nor does a low  $R^2$  necessarily mean poor predictability.
6. *No-intercept model.* The definition of  $R^2$  assumes that there is an intercept in the regression model. When the intercept is left out of the model, the definition of  $R^2$  changes dramatically. The fact that your  $R^2$  value increases when you remove the intercept from the regression model does not reflect an increase in the goodness of fit. Rather, it reflects a change in the underlying definition of  $R^2$ .
7. *Sample size.*  $R^2$  is highly sensitive to the number of observations. The smaller the sample size, the larger its value.

 **$R_{bar}^2$  (Adjusted  $R^2$ )**

$R^2$  varies directly with  $N$ , the sample size. In fact, when  $N = p$ ,  $R^2 = 1$ . Because  $R^2$  is so closely tied to the sample size, an adjusted  $R^2$  value, called  $\bar{R}^2$ , has been developed.  $\bar{R}^2$  was developed to minimize the impact of sample size. The formula for  $\bar{R}^2$  is

$$\bar{R}^2 = 1 - \frac{(N-1)(1-R^2)}{N-p-1}$$

---

## Testing Assumptions Using Residual Diagnostics

Evaluating the amount of departure in your data from each assumption is necessary to see if remedial action is necessary before the fitted results can be used. First, the types of plots and statistical analyses that are used to evaluate each assumption will be given. Second, each of the diagnostic values will be defined.

---

### Notation – Use of (j) and p

Several of these residual diagnostic statistics are based on the concept of studying what happens to various aspects of the regression analysis when each row is removed from the analysis. In what follows, we use the notation (*j*) to mean that observation *j* has been omitted from the analysis. Thus,  $b(j)$  means the value of *b* calculated without using observation *j*.

Some of the formulas depend on whether the intercept is fitted or not. We use *p* to indicate the number of regression parameters. When the intercept is fit, *p* will include the intercept.

---

### 1 – No Outliers

Outliers are observations that are poorly fit by the regression model. If outliers are influential, they will cause serious distortions in the regression calculations. Once an observation has been determined to be an outlier, it must be checked to see if it resulted from a mistake. If so, it must be corrected or omitted. However, if no mistake can be found, the outlier should not be discarded just because it is an outlier. Many scientific discoveries have been made because outliers, data points that were different from the norm, were studied more closely. Besides being caused by simple data-entry mistakes, outliers often suggest the presence of an important independent variable that has been ignored.

---

### 2 – Linear Regression Function - No Curvature

The relationship between *Y* and each *X* is assumed to be linear (straight-line). No mechanism for curvature is included in the model. Although scatter plots of *Y* versus each *X* can show curvature in the relationship, the best diagnostic tool is the scatter plot of the residual versus each *X*. If curvature is detected, the model must be modified to account for the curvature. This may mean adding a quadratic term, taking logarithms of *Y* or *X*, or some other appropriate transformation.

---

### 3 – Constant Variance

The errors are assumed to have constant variance across all values of *X*. If there are a lot of data ( $N > 100$ ), non-constant variance can be detected on the scatter plots of the residuals versus each *X*. However, the most direct diagnostic tool to evaluate this assumption is a scatter plot of the absolute values of the residuals versus each *X*. Often, the assumption is violated because the variance increases with *X*. This will show up as a ‘megaphone’ pattern on the scatter plot.

When non-constant variance is detected, a variance-stabilizing transformation such as the square-root or logarithm may be used. However, the best solution is probably to use weighted regression, with weights inversely proportional to the magnitude of the residuals.

---

### 4 – Independent Errors

The *Y*'s, and thus the errors, are assumed to be independent. This assumption is usually ignored unless there is a reason to think that it has been violated, such as when the observations were taken across time. An easy way to evaluate this assumption is a scatter plot of the residuals versus their sequence number (assuming that the data are arranged in time sequence order). This plot should show a relative random pattern.

## Multiple Regression – Basic

The Durbin-Watson statistic is used as a formal test for the presence of first-order serial correlation. A more comprehensive method of evaluation is to look at the autocorrelations of the residuals at various lags. Large autocorrelations are found by testing each using Fisher's  $z$  transformation. Although Fisher's  $z$  transformation is only approximate in the case of autocorrelations, it does provide a reasonable measuring stick with which to judge the size of the autocorrelations.

If independence is violated, confidence intervals and hypothesis tests are erroneous. Some remedial method that accounts for the lack of independence must be adopted, such as using first differences or the Cochrane-Orcutt procedure.

### Durbin-Watson Test

The Durbin-Watson test is often used to test for positive or negative, first-order, serial correlation. It is calculated as follows

$$DW = \frac{\sum_{j=2}^N (e_j - e_{j-1})^2}{\sum_{j=1}^N e_j^2}$$

The distribution of this test is difficult because it involves the  $X$  values. Originally, Durbin-Watson (1950, 1951) gave a pair of bounds to be used. However, there is a large range of 'inclusion' found when using these bounds. Instead of using these bounds, we calculate the exact probability using the beta distribution approximation suggested by Durbin-Watson (1951). This approximation has been shown to be accurate to three decimal places in most cases which is all that are needed for practical work.

---

## 5 – Normality of Residuals

The residuals are assumed to follow the normal probability distribution with zero mean and constant variance. This can be evaluated using a normal probability plot of the residuals. Also, normality tests are used to evaluate this assumption. The most popular of the five normality tests provided is the Shapiro-Wilk test.

Unfortunately, a breakdown in any of the other assumptions results in a departure from this assumption as well. Hence, you should investigate the other assumptions first, leaving this assumption until last.

---

## Data Structure

The data are entered in two or more columns. An example of data appropriate for this procedure is shown below. These data are from a study of the relationship of several variables with a person's I.Q. Fifteen people were studied. Each person's IQ was recorded along with scores on five different personality tests. The data are contained in the IQ dataset. We suggest that you open this database now so that you can follow along with the example.

## Multiple Regression – Basic

## IQ dataset

Test1	Test2	Test3	Test4	Test5	IQ
83	34	65	63	64	106
73	19	73	48	82	92
54	81	82	65	73	102
96	72	91	88	94	121
84	53	72	68	82	102
86	72	63	79	57	105
76	62	64	69	64	97
54	49	43	52	84	92
37	43	92	39	72	94
42	54	96	48	83	112
71	63	52	69	42	130
63	74	74	71	91	115
69	81	82	75	54	98
81	89	64	85	62	96
50	75	72	64	45	103

---

## Missing Values

Rows with missing values in the variables being analyzed are ignored. If data are present on a row for all but the dependent variable, a predicted value and confidence limits are generated for that row.

---

## Procedure Options

This section describes the options available in this procedure.

---

## Variables Tab

This panel specifies the variables used in the analysis.

---

### Variables

#### Y (Dependent Variable)

Specify the column containing the  $Y$  (dependent, response, or predicted) variable. This is the variable to be predicted by the independent ( $X$ ) variables.

The values in this column must be numeric. Any text values will be treated as missing values and ignored.

#### X's (Independent Variables)

Specify numeric independent (also called regressor, explanatory, or predictor) variables here. Numeric variables are those whose values are numeric and are at least ordinal. Categorical variables, even when coded with numbers, should be transformed into numeric contrasts or binary indicator variables.

If you want to create predicted values of  $Y$  for values of  $X$  not in your database, add the  $X$  new values as rows to the bottom of the database, with the value of  $Y$  blank. These rows will not be used during estimation phase, but predicted values will be generated for them on the reports.



## Example 1 – Basic Multiple Regression (All Reports)

This section presents an example of how to run a multiple regression analysis of the data presented earlier in this chapter. The data are in the IQ dataset. This example will run a regression of *IQ* on *Test1* through *Test5*. This regression program outputs many different reports and plots. For the purposes of annotating the output, all output is displayed.

You may follow along here by making the appropriate entries or load the completed template **Example 1** by clicking on Open Example Template from the File menu of the Multiple Regression - Basic window.

### 1 Open the IQ dataset.

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **IQ.NCSS**.
- Click **Open**.

### 2 Open the Multiple Regression - Basic window.

- Using the Analysis menu or the Procedure Navigator, find and select the **Multiple Regression - Basic** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 3 Specify the variables.

- On the Multiple Regression window, select the **Variables tab**.
- Set the **Y** box to **IQ**.
- Set the **X's** box to **Test1-Test5**.

### 4 Specify the reports.

- Select the **Reports tab**.
- Check all reports.

### 5 Specify the plots.

- Select the **Plots tab**.
- Check all plots.

### 6 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button.

## Run Summary Report

Item	Value	Rows	Value
Dependent Variable	IQ	Rows Processed	17
Number of Independent Variables	5	Rows Filtered Out	0
R <sup>2</sup>	0.3991	Rows with X's Missing	0
Adjusted R <sup>2</sup>	0.0652	Rows with Y Missing	2
Mean Square Error	113.4648	Rows used in Estimation	15
Completion Status	Normal Completion		

This report summarizes the multiple regression results. It presents the variables used, the number of rows used, and the basic results.

## Multiple Regression – Basic

 $R^2$ 

$R^2$ , officially known as the coefficient of determination, is defined as

$$R^2 = \frac{SS_{Model}}{SS_{Total(Adjusted)}}$$

$R^2$  is probably the most popular statistical measure of how well the regression model fits the data.  $R^2$  may be defined either as a ratio or a percentage. Since we use the ratio form, its values range from zero to one. A value of  $R^2$  near zero indicates no linear relationship between the  $Y$  and the  $X$ 's, while a value near one indicates a perfect linear fit. Although popular,  $R^2$  should not be used indiscriminately or interpreted without scatter plot support. Following are some qualifications on its interpretation:

1. *Additional independent variables.* It is possible to increase  $R^2$  by adding more independent variables, but the additional independent variables may actually cause an increase in the mean square error, an unfavorable situation. This case happens when your sample size is small.
2. *Range of the independent variables.*  $R^2$  is influenced by the range of each independent variable.  $R^2$  increases as the range of the  $X$ 's increases and decreases as the range of the  $X$ 's decreases.
3. *Slope magnitudes.*  $R^2$  does not measure the magnitude of the slopes.
4. *Linearity.*  $R^2$  does not measure the appropriateness of a linear model. It measures the strength of the linear component of the model. Suppose the relationship between  $x$  and  $Y$  was a perfect circle. The  $R^2$  value of this relationship would be zero.
5. *Predictability.* A large  $R^2$  does not necessarily mean high predictability, nor does a low  $R^2$  necessarily mean poor predictability.
6. *No-intercept model.* The definition of  $R^2$  assumes that there is an intercept in the regression model. When the intercept is left out of the model, the definition of  $R^2$  changes dramatically. The fact that your  $R^2$  value increases when you remove the intercept from the regression model does not reflect an increase in the goodness of fit. Rather, it reflects a change in the underlying meaning of  $R^2$ .
7. *Sample size.*  $R^2$  is highly sensitive to the number of observations. The smaller the sample size, the larger its value.

**Adjusted  $R^2$** 

This is an adjusted version of  $R^2$ . The adjustment seeks to remove the distortion due to a small sample size.

**Descriptive Statistics**

Variable	Count	Mean	Standard Deviation	Minimum	Maximum
Test1	15	67.93333	17.39239	37	96
Test2	15	61.4	19.39735	19	89
Test3	15	72.33334	14.73415	43	96
Test4	15	65.53333	13.95332	39	88
Test5	15	69.93333	16.15314	42	94
IQ	15	104.3333	11.0173	92	130

For each variable, the count, arithmetic mean, standard deviation, minimum, and maximum are computed. This report is particularly useful for checking that the correct variables were selected.

## Multiple Regression – Basic

## Correlation Matrix

	<b>Test1</b>	<b>Test2</b>	<b>Test3</b>	<b>Test4</b>
Test1	1.0000	0.1000	-0.2608	0.7539
Test2	0.1000	1.0000	0.0572	0.7196
Test3	-0.2608	0.0572	1.0000	-0.1409
Test4	0.7539	0.7196	-0.1409	1.0000
Test5	0.0140	-0.2814	0.3473	-0.1729
IQ	0.2256	0.2407	0.0741	0.3714
	<b>Test5</b>	<b>IQ</b>		
Test1	0.0140	0.2256		
Test2	-0.2814	0.2407		
Test3	0.3473	0.0741		
Test4	-0.1729	0.3714		
Test5	1.0000	-0.0581		
IQ	-0.0581	1.0000		

Pearson correlations are given for all variables. Outliers, non-normality, non-constant variance, and nonlinearities can all impact these correlations. Note that these correlations may differ from pair-wise correlations generated by the correlation matrix program because of the different ways the two programs treat rows with missing values. The method used here is row-wise deletion.

These correlation coefficients show which independent variables are highly correlated with the dependent variable and with each other. Independent variables that are highly correlated with one another may cause collinearity problems.

## Regression Coefficients T-Tests

Independent Variable	Regression Coefficient b(i)	Standard Error Sb(i)	Standardized Coefficient	T-Statistic to Test H0: $\beta(i)=0$	Prob Level	Reject H0 at 5%?
Intercept	85.24039	23.69514	0.0000	3.597	0.0058	Yes
Test1	-1.933571	1.029096	-3.0524	-1.879	0.0930	No
Test2	-1.659881	0.872896	-2.9224	-1.902	0.0897	No
Test3	0.1049543	0.219902	0.1404	0.477	0.6445	No
Test4	3.778377	1.834497	4.7853	2.060	0.0695	No
Test5	-0.04057754	0.2012205	-0.0595	-0.202	0.8447	No

This section reports the values and significance tests of the regression coefficients. Before using this report, check that the assumptions are reasonable. For instance, collinearity can cause the t-tests to give false results and the regression coefficients to be of the wrong magnitude or sign.

## Independent Variable

The names of the independent variables are listed here. The intercept is the value of the Y intercept.

Note that the name may become very long, especially for interaction terms. These long names may misalign the report. You can force the rest of the items to be printed on the next line by using the Skip Line After option in the Format tab. This should create a better looking report when the names are extra long.

## Regression Coefficient

The regression coefficients are the least squares estimates of the parameters. The value indicates how much change in Y occurs for a one-unit change in that particular X when the remaining X's are held constant. These coefficients are often called partial-regression coefficients since the effect of the other X's is removed. These coefficients are the values of  $b_0, b_1, \dots, b_p$ .

## Standard Error

The standard error of the regression coefficient,  $s_{b_j}$ , is the standard deviation of the estimate. It is used in hypothesis tests or confidence limits.

## Multiple Regression – Basic

### Standardized Coefficient

Standardized regression coefficients are the coefficients that would be obtained if you standardized the independent variables and the dependent variable. Here *standardizing* is defined as subtracting the mean and dividing by the standard deviation of a variable. A regression analysis on these standardized variables would yield these standardized coefficients.

When the independent variables have vastly different scales of measurement, this value provides a way of making comparisons among variables. The formula for the standardized regression coefficient is:

$$b_{j, std} = b_j \left( \frac{s_{X_j}}{s_Y} \right)$$

where  $s_Y$  and  $s_{X_j}$  are the standard deviations for the dependent variable and the  $j^{th}$  independent variable.

### T-Value to test Ho: B(i)=0

This is the t-test value for testing the hypothesis that  $\beta_j = 0$  versus the alternative that  $\beta_j \neq 0$  after removing the influence of all other  $X$ 's. This  $t$ -value has  $n-p-1$  degrees of freedom.

To test for a value other than zero, use the formula below. There is an easier way to test hypothesized values using confidence limits. See the discussion below under Confidence Limits. The formula for the  $t$ -test is

$$t_j = \frac{b_j - \beta_j^*}{s_{b_j}}$$

### Prob Level

This is the  $p$ -value for the significance test of the regression coefficient. The  $p$ -value is the probability that this  $t$ -statistic will take on a value at least as extreme as the actually observed value, assuming that the null hypothesis is true (i.e., the regression estimate is equal to zero). If the  $p$ -value is less than alpha, say 0.05, the null hypothesis of equality is rejected. This  $p$ -value is for a two-tail test.

### Reject H0 at 5%?

This is the conclusion reached about the null hypothesis. It will be either reject  $H_0$  at the 5% level of significance or not.

Note that the level of significance is specified in the Alpha of C.I.'s and Tests box on the Format tab panel.

### Note: The T-Value ...

This is the value of  $t_{1-\alpha/2, n-p-1}$  used to construct the confidence limits.

## Regression Coefficients Confidence Intervals

Independent Variable	Regression Coefficient b(i)	Standard Error Sb(i)	Lower 95% Conf. Limit of $\beta(i)$	Upper 95% Conf. Limit of $\beta(i)$
Intercept	85.24039	23.69514	31.63827	138.8425
Test1	-1.933571	1.029096	-4.261548	0.3944054
Test2	-1.659881	0.872896	-3.634509	0.3147467
Test3	0.1049543	0.219902	-0.3924985	0.6024072
Test4	3.778377	1.834497	-0.3715436	7.928297
Test5	-0.04057754	0.2012205	-0.49577	0.4146149

Note: The T-Value used to calculate these confidence limits was 2.262.

## Multiple Regression – Basic

### Independent Variable

The names of the independent variables are listed here. The intercept is the value of the  $Y$  intercept.

Note that the name may become very long, especially for interaction terms. These long names may misalign the report. You can force the rest of the items to be printed on the next line by using the Skip Line After option in the Format tab. This should create a better looking report when the names are extra long.

### Regression Coefficient

The regression coefficients are the least squares estimates of the parameters. The value indicates how much change in  $Y$  occurs for a one-unit change in  $x$  when the remaining  $X$ 's are held constant. These coefficients are often called partial-regression coefficients since the effect of the other  $X$ 's is removed. These coefficients are the values of  $b_0, b_1, \dots, b_p$ .

### Standard Error

The standard error of the regression coefficient,  $s_{b_j}$ , is the standard deviation of the estimate. It is used in hypothesis tests and confidence limits.

### Lower - Upper 95% C.L.

These are the lower and upper values of a  $100(1 - \alpha)\%$  interval estimate for  $\beta_j$  based on a  $t$ -distribution with  $n - p - 1$  degrees of freedom. This interval estimate assumes that the residuals for the regression model are normally distributed.

These confidence limits may be used for significance testing values of  $\beta_j$  other than zero. If a specific value is not within this interval, it is significantly different from that value. Note that these confidence limits are set up as if you are interested in each regression coefficient separately.

The formulas for the lower and upper confidence limits are:

$$b_j \pm t_{1-\alpha/2, n-p-1} s_{b_j}$$

## Estimated Equation

```

IQ =
85.2403846967439 - 1.93357123818932 * Test1 - 1.65988116961152 * Test2 + 0.104954325385776 * Test3
+ 3.77837667941384 * Test4 - 0.0405775409260279 * Test5
  
```

This is the least squares regression line presented in double precision. Besides showing the regression model in long form, it may be used as a transformation by copying and pasting it into the Transformation portion of the spreadsheet.

## Analysis of Variance

Source	DF	R <sup>2</sup> Lost If Term(s) Removed	Sum of Squares	Mean Square	F-Ratio	Prob Level
Intercept	1		163281.7	163281.7		
Model	5	0.3991	678.1504	135.6301	1.195	0.3835
Test1	1	0.2357	400.562	400.562	3.530	0.0930
Test2	1	0.2414	410.2892	410.2892	3.616	0.0897
Test3	1	0.0152	25.8466	25.8466	0.228	0.6445
Test4	1	0.2832	481.3241	481.3241	4.242	0.0695
Test5	1	0.0027	4.614109	4.614109	0.041	0.8447
Error	9	0.6009	1021.183	113.4648		
Total(Adjusted)	14		1699.333	121.381		

## Multiple Regression – Basic

This analysis of variance table provides a line for each term in the model. It is especially useful when you have categorical independent variables.

### Source

This is the term from the design model.

Note that the name may become very long, especially for interaction terms. These long names may misalign the report. You can force the rest of the items to be printed on the next line by using the Skip Line After option in the Format tab. This should create a better looking report when the names are extra long.

### DF

This is the number of degrees of freedom that the model is degrees of freedom is reduced when this term is removed from the model. This is the numerator degrees of freedom of the *F-test*.

### R<sup>2</sup> Lost if Term(s) Removed

This is the amount that  $R^2$  is reduced when this term is removed from the regression model.

### Sum of Squares

This is the amount that the model sum of squares that are reduced when this term is removed from the model.

### Mean Square

The mean square is the sum of squares divided by the degrees of freedom.

### F-Ratio

This is the *F*-statistic for testing the null hypothesis that all  $\beta_j$  associated with this term are zero. This *F*-statistic has *DF* and  $n-p-1$  degrees of freedom.

### Prob Level

This is the *p*-value for the above *F*-test. The *p*-value is the probability that the test statistic will take on a value at least as extreme as the observed value, assuming that the null hypothesis is true. If the *p*-value is less than  $\alpha$ , say 0.05, the null hypothesis is rejected. If the *p*-value is greater than  $\alpha$ , then the null hypothesis is accepted.

---

## Normality Tests

Normality Tests			
Test Name	Test Statistic to Test H0: Normal	Prob Level	Reject H0 at 20%?
Shapiro Wilk	0.908	0.1243	Yes
Anderson Darling	0.458	0.2639	No
D'Agostino Skewness	2.033	0.0421	Yes
D'Agostino Kurtosis	1.580	0.1141	Yes
D'Agostino Omnibus	6.629	0.0364	Yes

This report gives the results of applying several normality tests to the residuals. The Shapiro-Wilk test is probably the most popular, so it is given first. These tests are discussed in detail in the Normality Test section of the Descriptive Statistics procedure.

## Serial-Correlation and Durbin-Watson Test

### Serial Correlation of Residuals

Lag	Serial Correlation	Lag	Serial Correlation	Lag	Serial Correlation
1	0.4529	9	-0.2769	17	0.0000
2	-0.2507	10	-0.2287	18	0.0000
3	-0.5518	11	-0.0197	19	0.0000
4	-0.3999	12	0.0669	20	0.0000
5	0.0780	13	0.0000	21	0.0000
6	0.2956	14	0.0000	22	0.0000
7	0.1985	15	0.0000	23	0.0000
8	-0.0016	16	0.0000	24	0.0000

Above serial correlations significant if their absolute values are greater than 0.5164

### Durbin-Watson Test for Serial Correlation

Parameter	Test Statistic to Test H0: $\rho(1) = 0$ ?	Reject H0 at 20%?
Durbin-Watson Value	1.001	
Prob. Level: Positive Serial Correlation	0.0072	Yes
Prob. Level: Negative Serial Correlation	0.9549	No

This section reports the autocorrelation structure of the residuals. Of course, this report is only useful if the data represent a time series.

### Lag and Correlation

The lag,  $k$ , is the number of periods (rows) back. The correlation here is the sample autocorrelation coefficient of lag  $k$ . It is computed as:

$$r_k = \frac{\sum e_{i-k} e_i}{\sum e_i^2} \quad \text{for } k = 1, 2, \dots, 24$$

To test the null hypothesis that  $\rho_k = 0$  at a 5% level of significance with a large-sample normal approximation, reject when the absolute value of the autocorrelation coefficient,  $|r_k|$ , is greater than two over the square root of  $N$ .

### Durbin-Watson Value

The Durbin-Watson test is often used to test for positive or negative, first-order, serial correlation. It is calculated as follows

$$r_k = \frac{\sum e_{i-k} e_i}{\sum e_i^2} \quad \text{for } k = 1, 2, \dots, 24$$

The distribution of this test is mathematically difficult because it involves the  $X$  values. Originally, Durbin-Watson (1950, 1951) gave a pair of bounds to be used. However, there is a large range of indecision that can be found when using these bounds. Instead of using these bounds, NCSS calculates the exact probability using the beta distribution approximation suggested by Durbin-Watson (1951). This approximation has been shown to be accurate to three decimal places in most cases.

## Multiple Regression – Basic

## Multicollinearity Report

Independent Variable	Variance Inflation Factor	R <sup>2</sup> Versus Other I.V.'s	Tolerance	Diagonal of X'X Inverse
Test1	39.5273	0.9747	0.0253	0.009333631
Test2	35.3734	0.9717	0.0283	0.006715277
Test3	1.2953	0.2280	0.7720	0.0004261841
Test4	80.8456	0.9876	0.0124	0.02966012
Test5	1.3035	0.2329	0.7671	0.0003568483

This report provides information useful in assessing the amount of multicollinearity in your data.

### Variance Inflation Factor

The variance inflation factor (*VIF*) is a measure of multicollinearity. It is the reciprocal of  $1 - R_X^2$ , where  $R_X^2$  is the  $R^2$  obtained when this variable is regressed on the remaining IV's. A *VIF* of 10 or more for large data sets indicates a collinearity problem since the  $R_X^2$  with the remaining IV's is 90 percent. For small data sets, even *VIF*'s of 5 or more can signify collinearity. Variables with a high *VIF* are candidates for exclusion from the model.

$$VIF_j = \frac{1}{1 - R_j^2}$$

### R<sup>2</sup> Versus Other IV's

$R_X^2$  is the  $R^2$  obtained when this variable is regressed on the remaining independent variables. A high  $R_X^2$  indicates a lot of overlap in explaining the variation among the remaining independent variables.

### Tolerance

Tolerance is just  $1 - R_X^2$ , the denominator of the variance inflation factor.

### Diagonal of X'X Inverse

The X'X inverse is an important matrix in regression. This is the  $j^{\text{th}}$  row and  $j^{\text{th}}$  column element of this matrix.

## Predicted Values with Confidence Limits of Means

Row	Actual IQ	Predicted IQ	Standard Error Of Predicted	95% Lower Conf. Limit Of Mean	95% Upper Conf. Limit Of Mean
1	106	110.5808	7.156587	94.39149	126.7701
2	92	98.24833	7.075774	82.24182	114.2548
3	102	97.61574	6.222865	83.53864	111.6928
4	121	118.3398	8.686609	98.68933	137.9903
5	102	96.00567	6.369277	81.59736	110.414
6	105	102.2328	5.433011	89.94245	114.5231
7	97	100.2044	4.099899	90.92982	109.4791
8	92	97.07347	9.098985	76.49014	117.6568
9	94	96.41426	7.088691	80.37853	112.45
10	112	102.4666	6.351596	88.09826	116.8349
11	130	107.8457	6.464094	93.22288	122.4685
12	115	112.933	7.331204	96.34866	129.5173
13	98	107.1669	5.338698	95.08994	119.2439
14	96	106.255	5.531748	93.74129	118.7687
15	103	111.6176	7.100195	95.55581	127.6793
16		97.70544	7.031275	81.79959	113.6113
17		100.1984	4.305268	90.45926	109.9376

Confidence intervals for the mean response of *Y* given specific levels for the IV's are provided here. It is important to note that violations of any regression assumptions will invalidate these interval estimates.



## Multiple Regression – Basic

**Actual**

This is the actual value of  $Y$ .

**Predicted**

The predicted value of  $Y$ . It is predicted using the values of the IV's for this row. If the input data had all IV values but no value for  $Y$ , the predicted value is still provided.

**Standard Error of Predicted**

This is the standard error of the mean response for the specified values of the IV's. Note that this value is not constant for all IV's values. In fact, it is a minimum at the average value of each IV.

**Lower 95% C.L. of Mean**

This is the lower limit of a 95% confidence interval estimate of the mean of  $Y$  for this observation.

**Upper 95% C.L. of Mean**

This is the upper limit of a 95% confidence interval estimate of the mean of  $Y$  for this observation. Note that you set the alpha level.

---

**Predicted Values with Prediction Limits of Individuals**

Row	Actual IQ	Predicted IQ	Standard Error Of Predicted	95% Lower Pred. Limit Of Individual	95% Upper Pred. Limit Of Individual
1	106	110.5808	12.83283	81.55093	139.6107
2	92	98.24833	12.78794	69.32001	127.1767
3	102	97.61574	12.33648	69.70868	125.5228
4	121	118.3398	13.74489	87.2467	149.4329
5	102	96.00567	12.41098	67.93008	124.0813
6	105	102.2328	11.95752	75.18298	129.2826
7	97	100.2044	11.41376	74.38472	126.0242
8	92	97.07347	14.00915	65.38257	128.7644
9	94	96.41426	12.79509	67.46976	125.3588
10	112	102.4666	12.40192	74.41148	130.5217
11	130	107.8457	12.45991	79.6594	136.0319
12	115	112.933	12.93102	83.68099	142.185
13	98	107.1669	11.91497	80.21338	134.1204
14	96	106.255	12.00271	79.10297	133.407
15	103	111.6176	12.80147	82.65864	140.5765
16		97.70544	12.76337	68.83269	126.5782
17		100.1984	11.48913	74.20823	126.1887

A prediction interval for the individual response of  $Y$  given specific values of the IV's is provided here for each row.

**Actual**

This is the actual value of  $Y$ .

**Predicted**

The predicted value of  $Y$ . It is predicted using the levels of the IV's for this row. If the input data had all values of the IV's but no value for  $Y$ , a predicted value is provided.

**Standard Error of Predicted**

This is the standard deviation of the mean response for the specified levels of the IV's. Note that this value is not constant for all IV's. In fact, it is a minimum at the average value of each IV.

**Lower 95% Pred. Limit of Individual**

This is the lower limit of a 95% prediction interval of the individual value of  $Y$  for the values of the IV's for this observation.

## Multiple Regression – Basic

## Upper 95% Pred. Limit of Individual

This is the upper limit of a 95% prediction interval of the individual value of  $Y$  for the values of the IV's for this observation. Note that you set the alpha level.

## Residual Report

Row	Actual IQ	Predicted IQ	Residual	Absolute Percent Error	Sqrt(MSE) Without This Row
1	106	110.5808	-4.580812	4.322	11.08453
2	92	98.24833	-6.24833	6.792	10.90476
3	102	97.61574	4.384259	4.298	11.13554
4	121	118.3398	2.660196	2.199	11.18066
5	102	96.00567	5.994334	5.877	10.98439
6	105	102.2328	2.767226	2.635	11.24073
7	97	100.2044	-3.204443	3.304	11.23126
8	92	97.07347	-5.073471	5.515	10.75852
9	94	96.41426	-2.414264	2.568	11.24011
10	112	102.4666	9.53343	8.512	10.489
11	130	107.8457	22.15433	17.042	5.525609
12	115	112.933	2.067002	1.797	11.25314
13	98	107.1669	-9.166913	9.354	10.65928
14	96	106.255	-10.25498	10.682	10.47129
15	103	111.6176	-8.617564	8.367	10.53295
16		97.70544			
17		100.1984			

This section reports on the sample residuals, or  $e_i$ 's.

**Actual**

This is the actual value of  $Y$ .

**Predicted**

The predicted value of  $Y$  using the values of the IV's given on this row.

**Residual**

This is the error in the predicted value. It is equal to the *Actual* minus the *Predicted*.

**Absolute Percent Error**

This is percentage that the absolute value of the *Residual* is of the *Actual* value. Scrutinize rows with the large percent errors.

**Sqrt(MSE) Without This Row**

This is the value of the square root of the mean square error that is obtained if this row is deleted. A perusal of this statistic for all observations will highlight observations that have an inflationary impact on mean square error and could be outliers.

## Graphic Residual Analysis

The residuals can be graphically analyzed in numerous ways. Three types of residuals are graphically analyzed here: residuals, student residuals, and partial residuals. For certain, the regression analyst should examine all of the basic residual graphs: the histogram, the density trace, the normal probability plot, the serial correlation plots, the scatter plot of the residuals versus the sequence of the observations, the scatter plot of the residuals versus the predicted value of the dependent variable, and the scatter plot of the residuals versus each of the independent variables.

## Multiple Regression – Basic

For the basic scatter plots of residuals versus either the predicted values of  $Y$  or the independent variables, Hoaglin (1983) explains that there are several patterns to look for. You should note that these patterns are very difficult, if not impossible, to recognize for small data sets.

### Point Cloud

A point cloud, basically in the shape of a rectangle or a horizontal band, would indicate no relationship between the residuals and the variable plotted against them. This is the preferred condition.

### Wedge

An increasing or decreasing wedge would be evidence that there is increasing or decreasing (nonconstant) variation. A transformation of  $Y$  may correct the problem, or weighted least squares may be needed.

### Bowtie

This is similar to the wedge above in that the residual plot shows a decreasing wedge in one direction while simultaneously having an increasing wedge in the other direction. A transformation of  $Y$  may correct the problem, or weighted least squares may be needed.

### Sloping Band

This kind of residual plot suggests adding a linear version of the independent variable to the model.

### Curved Band

This kind of residual plot may be indicative of a nonlinear relationship between  $Y$  and the independent variables that was not accounted for. The solution might be to use a transformation on  $Y$  to create a linear relationship with the  $X$ 's. Another possibility might be to add quadratic or cubic terms of a particular independent variable.

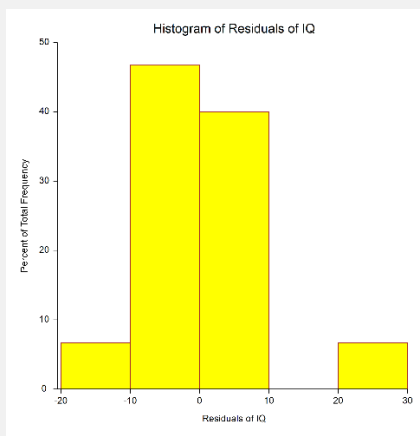
### Curved Band with Increasing or Decreasing Variability

This residual plot is really a combination of the wedge and the curved band. It too must be avoided.

---

## Histogram

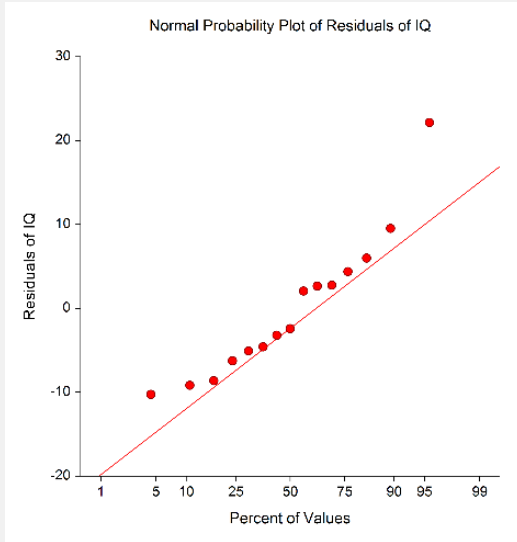
The purpose of the histogram and density trace of the residuals is to evaluate whether they are normally distributed. A dot plot is also given that highlights the distribution of points in each bin of the histogram. Unless you have a large sample size, it is best not to rely on the histogram for visually evaluating normality of the residuals. The better choice would be the normal probability plot.



## Probability Plot of Residuals

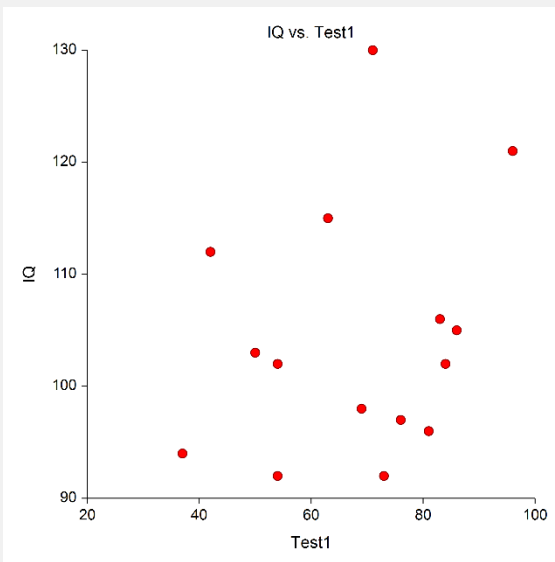
If the residuals are normally distributed, the data points of the normal probability plot will fall along a straight line through the origin with a slope of 1.0. Major deviations from this ideal picture reflect departures from normality. Stragglers at either end of the normal probability plot indicate outliers, curvature at both ends of the plot indicates long or short distributional tails, convex or concave curvature indicates a lack of symmetry, and gaps or plateaus or segmentation in the normal probability plot may require a closer examination of the data or model. Of course, use of this graphic tool with very small sample sizes is not recommended.

If the residuals are not normally distributed, then the t-tests on regression coefficients, the F-tests, and any interval estimates are not valid. This is a critical assumption to check.



## Plots of Y versus each IV

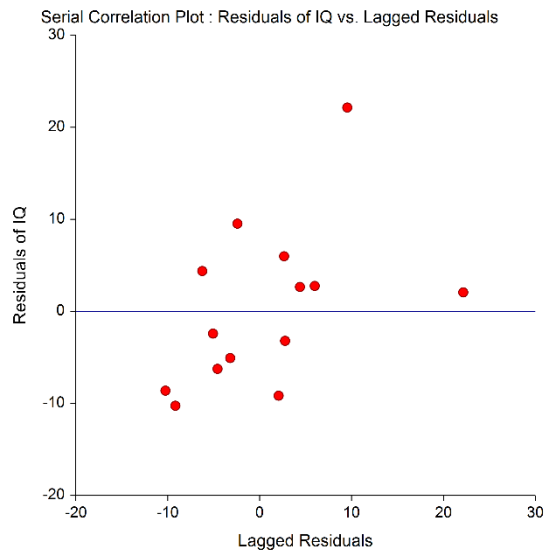
Actually, a regression analysis should always begin with a plot of  $Y$  versus each IV. These plots often show outliers, curvilinear relationships, and other anomalies.



## Serial Correlation of Residuals Plot

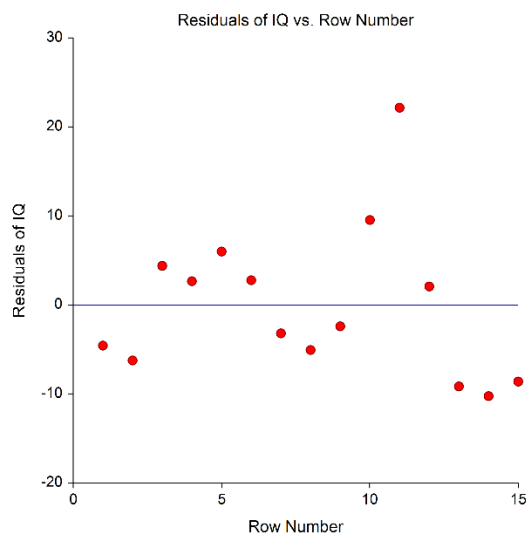
This plot is only useful if your data represent a time series. This is a scatter plot of the  $j^{\text{th}}$  residual versus the  $j^{\text{th}}-1$  residual. The purpose of this plot is to check for first-order autocorrelation.

You would like to see a random pattern of these plotted residuals, i.e., a rectangular or uniform distribution. A strong positive or negative trend would indicate a need to redefine the model with some type of autocorrelation component. Positive autocorrelation or serial correlation means that the residual in time period  $j$  tends to have the same sign as the residual in time period  $(j-1)$ . On the other hand, a strong negative autocorrelation means that the residual in time period  $j$  tends to have the opposite sign as the residual in time period  $(j-1)$ . Be sure to check the Durbin-Watson statistic.



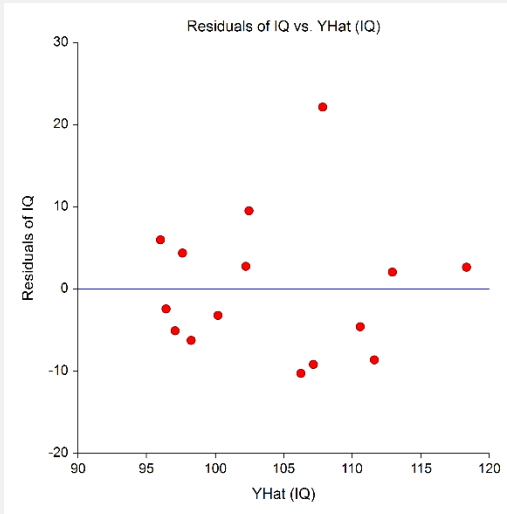
## Sequence Plot

Sequence plots may be useful in finding variables that are not accounted for by the regression equation. They are especially useful if the data were taken over time.



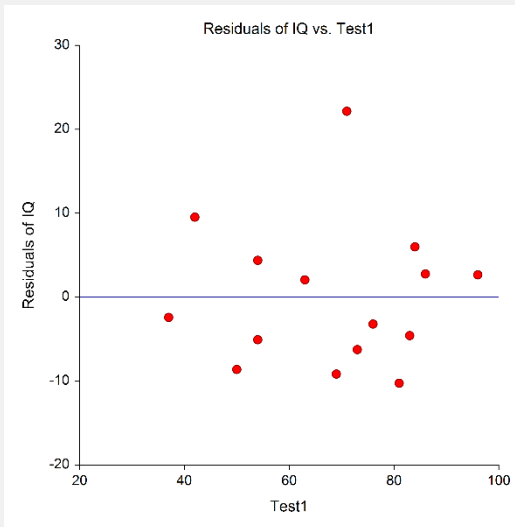
## Residual vs Predicted Plot

This plot should always be examined. The preferred pattern to look for is a point cloud or a horizontal band. A wedge or bowtie pattern is an indicator of non-constant variance, a violation of a critical regression assumption. The sloping or curved band signifies inadequate specification of the model. The sloping band with increasing or decreasing variability suggests non-constant variance and inadequate specification of the model.



## Residual vs Predictor(s) Plot

This is a scatter plot of the residuals versus each independent variable. Again, the preferred pattern is a rectangular shape or point cloud. Any other nonrandom pattern may require a redefining of the regression model.



---

## Example 2 – Multiple Regression with a Categorical Variable

This section presents an example of how to run a multiple regression analysis with a binary indicator variable. The data for this example are contained in the dataset called Salary. The dataset consists of yearly salaries (*Salary*) of 60 individuals working in the same profession, the number of months each has worked (*Months*), and an indicator variable (*Size*) specifying the size of the company at which the individual is currently employed. A second indicator variable (*Size\_binary*) translates the size categories into a numeric variable for use in this procedure with “0” for *Size* = “Small” and “1” for *Size* = “Large” (The general Multiple Regression procedure can analyze categorical variables directly, but this procedure requires all independent variables to be numeric). The goal of this analysis is to determine whether the amount of experience actually translates into higher wages for these individuals and whether the size of a company plays a significant part in the relationship.

You may follow along here by making the appropriate entries or load the completed template **Example 2** by clicking on Open Example Template from the File menu of the Multiple Regression - Basic window.

### 1 Open the Salary dataset.

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **Salary.NCSS**.
- Click **Open**.

### 2 Open the Multiple Regression - Basic window.

- Using the Analysis menu or the Procedure Navigator, find and select the **Multiple Regression - Basic** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 3 Specify the variables.

- On the Multiple Regression window, select the **Variables tab**.
- Set the **Y** box to **Salary**.
- Set the **X's** box to **Months, Size\_binary**.

### 4 Specify the reports.

- Select the **Reports tab**.
- Check **Coefficient T-Tests**.

### 5 Specify the plots.

- Select the **Plots tab**.
- Check **Y vs X**.

### 6 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button.

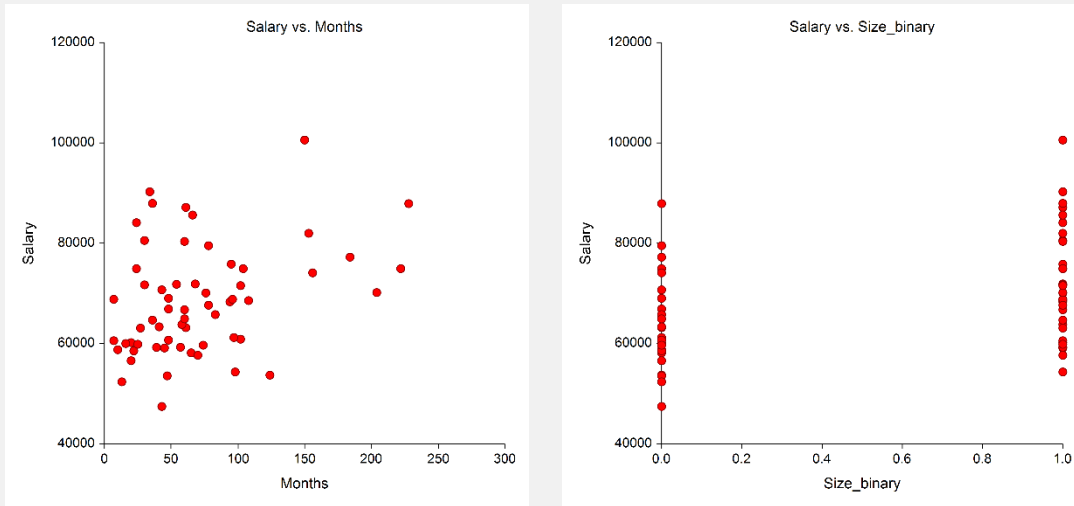
Multiple Regression – Basic

Output

Regression Coefficients T-Tests

Independent Variable	Regression Coefficient b(i)	Standard Error Sb(i)	Standardized Coefficient	T-Statistic to Test H0: $\beta(i)=0$	Prob Level	Reject H0 at 5%?
Intercept	57596.64	2627.511	0.0000	21.921	0.0000	Yes
Months	82.48019	23.77615	0.3943	3.469	0.0010	Yes
Size_binary	8494.26	2473.689	0.3903	3.434	0.0011	Yes

Plots Section



The results indicate that there is a significant positive relationship between an individual’s experience and their salary (p-value = 0.0010), as you would expect. On average, a person’s yearly salary increases by \$82.48 for every month on the job. The size of the company has an impact as well (p-value = 0.0011) – on average workers at larger companies make about \$8,494.26 more than their counterparts at smaller companies after accounting for the amount of experience. These relationships can be seen in the two scatterplots.



---

## Example 3 – Fitting a Curvilinear Relationship using Multiple Regression

This section presents an example of how to fit a quadratic relationship using multiple regression. The data for this example are contained in the MRPoly dataset. The dataset contains variables Y, X, and X2. The variable X2 was created by squaring the values in X.

You may follow along here by making the appropriate entries or load the completed template **Example 3a** by clicking on Open Example Template from the File menu of the Multiple Regression - Basic window.

### 1 Open the MRPoly dataset.

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **MRPoly.NCSS**.
- Click **Open**.

### 2 Open the Multiple Regression - Basic window.

- Using the Analysis menu or the Procedure Navigator, find and select the **Multiple Regression - Basic** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

### 3 Specify the variables.

- On the Multiple Regression window, select the **Variables tab**.
- Set the **Y** box to **Y**.
- Set the **X's** box to **X**.

### 4 Specify the reports.

- Select the **Reports tab**.
- Check **Run Summary**, **Descriptive Statistics**, and **Coefficient T-Tests**.

### 5 Specify the plots.

- Select the **Plots tab**.
- Check **Y vs X** and **Residuals vs X**.

### 6 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button.

## Multiple Regression – Basic

## Output

## Run Summary Report

Item	Value	Rows	Value
Dependent Variable	Y	Rows Processed	150
Number of Independent Variables	1	Rows Filtered Out	0
R <sup>2</sup>	0.8781	Rows with X's Missing	0
Adjusted R <sup>2</sup>	0.8773	Rows with Y Missing	0
Mean Square Error	905732.7	Rows used in Estimation	150
Completion Status	Normal Completion		

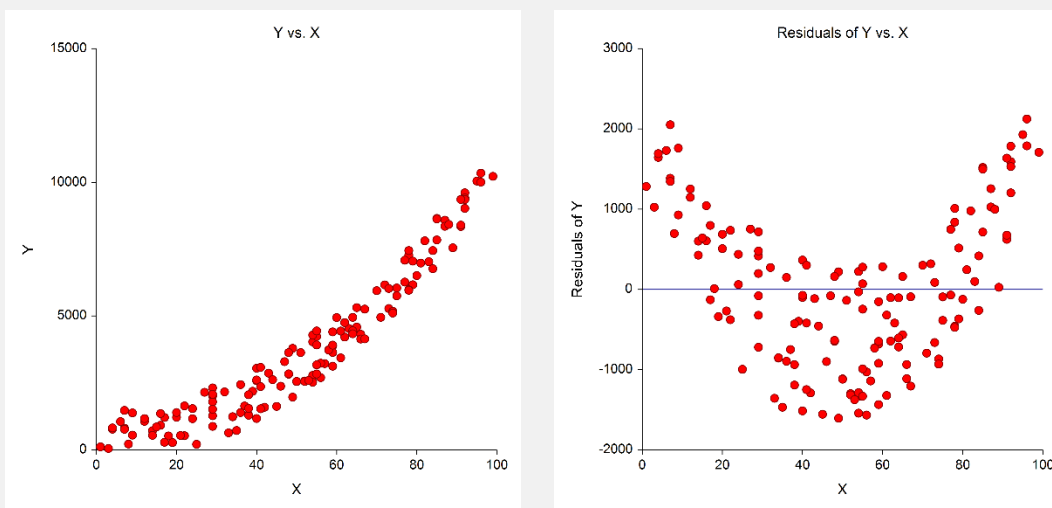
## Descriptive Statistics

Variable	Count	Mean	Standard Deviation	Minimum	Maximum
X	150	51.08667	25.75854	1	99
Y	150	3783.813	2716.466	56	10346

## Regression Coefficients T-Tests

Independent Variable	Regression Coefficient b(i)	Standard Error Sb(i)	Standardized Coefficient	T-Statistic to Test H0: $\beta(i)=0$	Prob Level	Reject H0 at 5%?
Intercept	-1264.638	173.0566	0.0000	-7.308	0.0000	Yes
X	98.8213	3.026813	0.9371	32.649	0.0000	Yes

## Plots Section



The results indicate a significant relationship between Y and X (p-value = 0.0000) with an R<sup>2</sup> value of 0.8781. However, the Y vs. X and Residual plots indicate that there is a non-linear component that should be accounted for. The U-shaped curve in the residual plot indicates that the model should also involve a quadratic term. The variable X2 in the dataset contains the squared values of X, so we'll add that now to the model. Make the following changes or load the completed template **Example 3b** by clicking on Open Example Template from the File menu of the Multiple Regression - Basic window.

## 7 Specify the variables.

- On the Multiple Regression window, select the **Variables tab**.
- Set the **X's** box to **X, X2**.

## 8 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button.

## Multiple Regression – Basic

# Output

### Run Summary Report

Item	Value	Rows	Value
Dependent Variable	Y	Rows Processed	150
Number of Independent Variables	2	Rows Filtered Out	0
R <sup>2</sup>	0.9593	Rows with X's Missing	0
Adjusted R <sup>2</sup>	0.9587	Rows with Y Missing	0
Mean Square Error	304684.4	Rows used in Estimation	150
Completion Status	Normal Completion		

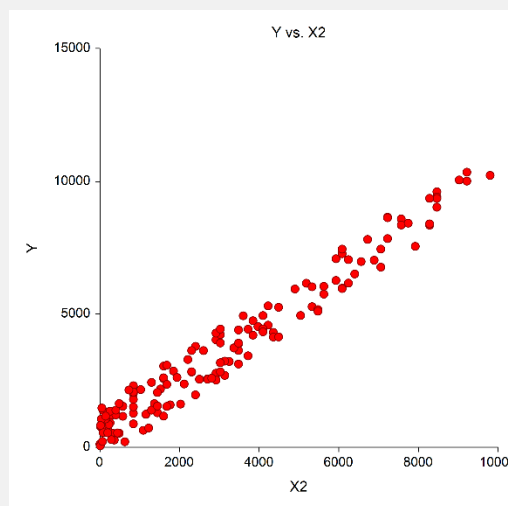
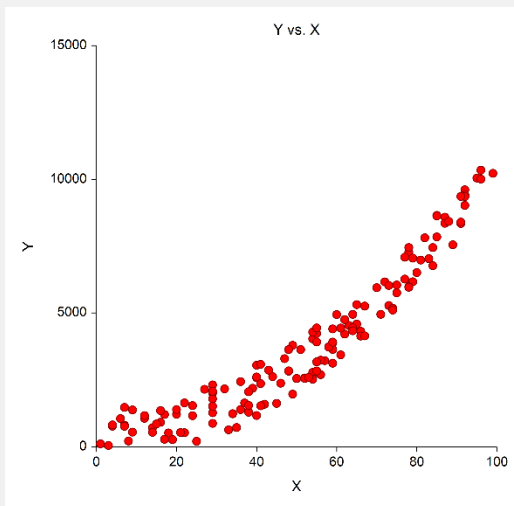
### Descriptive Statistics

Variable	Count	Mean	Standard Deviation	Minimum	Maximum
X	150	51.08667	25.75854	1	99
X2	150	3268.927	2656.684	1	9801
Y	150	3783.813	2716.466	56	10346

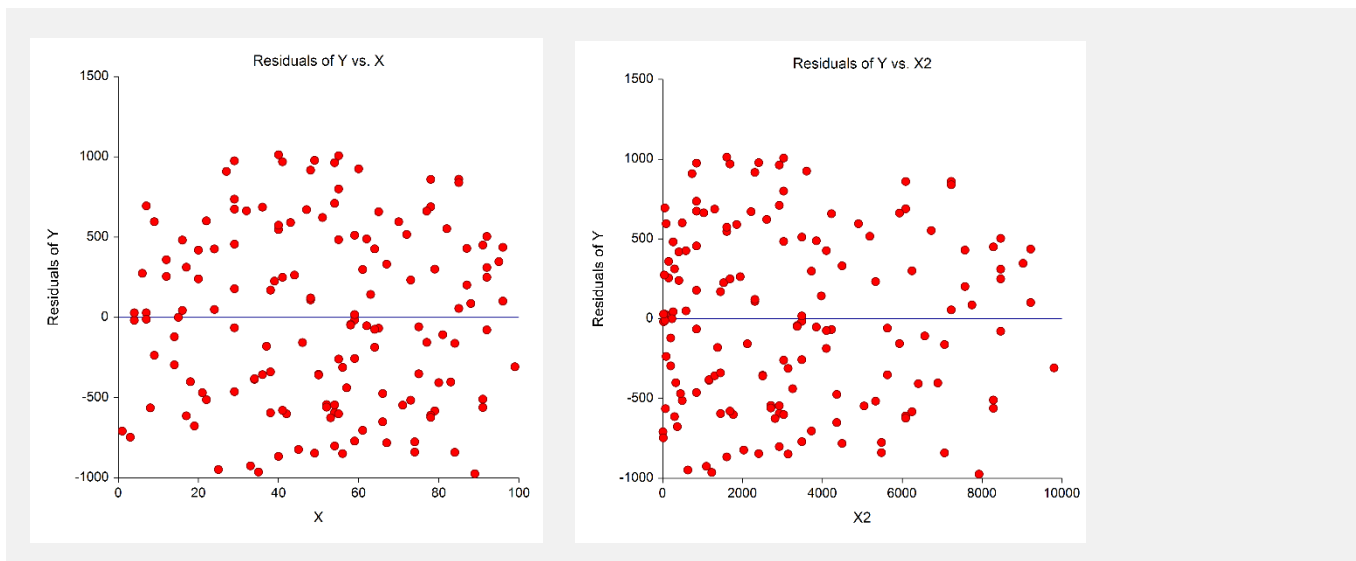
### Regression Coefficients T-Tests

Independent Variable	Regression Coefficient b(i)	Standard Error Sb(i)	Standardized Coefficient	T-Statistic to Test H0: $\beta(i)=0$	Prob Level	Reject H0 at 5%?
Intercept	839.8864	158.7225	0.0000	5.292	0.0000	Yes
X	-16.02258	6.935588	-0.1519	-2.310	0.0223	Yes
X2	1.150979	0.06724572	1.1256	17.116	0.0000	Yes

### Plots Section



## Multiple Regression – Basic



The results indicate that the addition of the squared X term provides a better fit with an  $R^2$  value of 0.9593. Both X and  $X^2$  are significant with p-values  $< 0.05$ . The residual plots demonstrate random scatter as would be expected for a well-fitting model.