Chapter 483

Quadratic Programming

Introduction

Quadratic programming maximizes (or minimizes) a quadratic objective function subject to one or more constraints. The technique finds broad use in operations research and is occasionally of use in statistical work.

The mathematical representation of the quadratic programming (QP) problem is

Maximize

$$z = \boldsymbol{CX} + \frac{1}{2}\boldsymbol{X'HX} \quad \text{or} \quad z = \boldsymbol{CX} + \boldsymbol{X'DX}$$

subject to

$$\mathbf{AX} \le \mathbf{b},\ \mathbf{X} \ge \mathbf{0}$$

where

$$\boldsymbol{X} = (x_1, x_2, \ldots, x_n)'$$

$$\boldsymbol{C} = (c_1, c_2, \ldots, c_n)$$

$$\boldsymbol{b} = (b_1, b_2, \ldots, b_m)'$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & d_{nn} \end{bmatrix}$$

The symmetric matrix ***H*** is often called the Hessian. The upper-triangular matrix ***D*** is constructed from ***H*** using

$$\boldsymbol{D} = \begin{pmatrix} 2h_{11} & \cdots & h_{1i} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 2h_{ii} & \cdots & h_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 2h_{nn} \end{pmatrix}$$

The x_i's are the *decision variables* (the unknowns), the first equation is called the *objective function* and the *m* inequalities (and equalities) are called *constraints*. The constraint bounds, the b_i's, are often called *right-hand sides* (RHS).

NCSS solves a particular quadratic program using a primal active set method available in the *Extreme Optimization* mathematical subroutine package.

Example

We will solve the following problem using **NCSS**:

Minimize

$$z = x_1 - 2x_2 + 4x_3 + x_1^2 + 2x_2^2 + 3x_3^2 + x_1 x_3$$

subject to

$$3x_1 + 4x_2 - 2x_3 \leq 10$$

$$-3x_1 + 2x_2 + x_3 \geq 2$$

$$2x_1 + 3x_2 + 4x_3 = 5$$

$$0 \leq x_1 \leq 5$$

$$1 \leq x_2 \leq 5$$

$$0 \leq x_3 \leq 5$$

The solution (see Example 1 below) is x_1 = 0.290, x_2 = 1.413, and x_3 = 0.045, which results in z = 1.741.

Data Structure

This technique requires a special data format which will be discussed under the *Specifications* tab. Here is the way the above example would be entered. It is stored in the dataset QP.

QP dataset

Type	Logic	RHS	X1	X2	X3	D1	D2	D3
O			1	-2	4	1	0	1
C	<	10	3	4	-2		2	0
C	>	2	-3	2	1			3
C	=	5	2	3	4			
L			0	1	0			
U			5	5	5			

Example 1 – Quadratic Programming

This section presents an example of how to run the data presented in the example given above. The data are contained in the QP database. Here is the specification of the problem.

Minimize

$$z = x_1 - 2x_2 + 4x_3 + x_1^2 + 2x_2^2 + 3x_3^2 + x_1 x_3$$

subject to

$$3x_1 + 4x_2 - 2x_3 \leq 10$$

$$-3x_1 + 2x_2 + x_3 \geq 2$$

$$2x_1 + 3x_2 + 4x_3 = 5$$

$$0 \leq x_1 \leq 5$$

$$1 \leq x_2 \leq 5$$

$$0 \leq x_3 \leq 5$$

Setup

To run this example, complete the following steps:

1 Open the QP example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **QP** and click **OK**.

2 Specify the Quadratic Programming procedure options

- Find and open the **Quadratic Programming** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 1** settings template. To load this template, click **Open Example Template** in the Help Center or File menu.

Option	Value
Specifications Tab	
Type of Optimum	**Minimum**
Row Type Column	**Type**
Variables Columns	**X1-X3**
Labels of Constraints Column	**CLabel**
Input Type of Quadratic Terms	**Quadratic Coefficients**
Quadratic Coefficients Columns	**D1-D3**
Logic Column	**Logic**
Constraint Bounds (RHS) Column	**RHS**

3 Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

Objective Function and Solution for Minimum

Objective Function and Solution for Minimum

Variable	Objective Function Coefficient	Value at Minimum	Lower Bound	Upper Bound
X1	1.0	0.290	0.0	5.0
X2	-2.0	1.413	1.0	5.0
X3	4.0	0.045	0.0	5.0

Minimum of Objective Function 1.741

Solution Status: The optimization model is optimal.

This report lists the linear portion of the objective function coefficients, the values of the variables at the minimum (that is, the solution), and the lower and upper bonds if specified. It also shows the value of the objective function at the solution as well as the status of the algorithm when it terminated.

Constraints

Constraints

Label, Logic	X1	X2	X3	RHS
Con1, ≤	3.0	4.0	-2.0	10.0
Con2, ≥	-3.0	2.0	1.0	2.0
Con3, =	2.0	3.0	4.0	5.0

This report presents the coefficients of the constraints as they were input.

Values of Constraints at Solution for Minimum

Values of Constraints at Solution for Minimum

Label, Logic	RHS	RHS at Solution
Con1, ≤	10.0	6.435
Con2, ≥	2.0	2.000
Con3, =	5.0	5.000

This report presents the right hand side of each constraint along with its value at the optimal values of the variables.

Hessian Matrix

Hessian Matrix

Variables	X1	X2	X3
X1	2.0	0.0	1.0
X2	0.0	4.0	0.0
X3	1.0	0.0	6.0

This report shows the Hessian matrix calculated from the D matrix that was input.

Quadratic Portion of the Objective Function

Quadratic Portion of the Objective Function

Variables	X1	X2	X3
X1	1.0	0.0	1.0
X2		2.0	0.0
X3			3.0

This report shows the coefficients of the quadratic portion of the objective function presented in matrix format.