Chapter 408

Confidence Intervals for Exponential Reliability

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the reliability (proportion surviving up to time t) when the survival time follows an exponential distribution. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Nelson (1982) and Mathews (2010). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta} \exp(-t/\theta), \ \ t \geq 0$$

where θ is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate, λ which is equal to $1/\theta$. In this case, the density is

$$f(t) = \lambda \exp(-\lambda t), \ \ t \geq 0.$$

The cumulative exponential (failure) distribution is

$$F(t) = 1 - \exp(-t/\theta), \ \ t \geq 0.$$

The survival or **reliability** function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution is simply

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of θ based on observing E failures in N items tested is

$$\bar{t} = \frac{1}{E} \sum_{k=1}^{N} t_k$$

where t_k is the amount of time that the kth item was under test, whether the event of interest was observed in that item or not.

An exact $100(1 - \alpha)$ % confidence interval for the $R(t)$ is given by

$$P\{LCL \le R(t) \le UCL\} = 1 - \alpha$$

where

$$LCL = \exp\left(-t\,\frac{\chi^2_{1-\frac{\alpha}{2}}}{2E\bar{t}}\right)$$

$$UCL = \exp\left(-t\,\frac{\chi^2_{\frac{\alpha}{2}}}{2E\bar{t}}\right)$$

Here, χ^2_φ is the value of the chi-square random variate with $2E$ degrees of freedom that has probability φ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α.

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL.$$

This equation can be used to find E, α, or the width.

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of n items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

This option specifies the parameter to be solved for from the other parameters.

One-Sided or Two-Sided Interval

Interval Type

Specify whether the confidence interval for the population correlation is two-sided or one-sided. A one-sided interval is often called a **confidence bound** rather than a confidence interval because it only has one limit.

Two-Sided

The two-sided confidence interval is defined by two limits: an upper confidence limit (UCL) and a lower confidence limit (LCL).

These limits are constructed so that the designated proportion (confidence level) of such intervals will include the true population value.

Upper One-Sided

The upper confidence interval (or bound) is defined by a limit above the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value below it.

Lower One-Sided

The lower confidence interval (or bound) is defined by a limit below the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value above it.

Confidence

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples with E failures are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Often, the values 0.95 or 0.99 are used. You can enter single values or a range of values such as *0.90, 0.95* or *0.90 to 0.99 by 0.01*.

Number of Events and Censoring

E (Number of Events)

Enter a value for the number of events, E. This is the number of subjects that exhibit the event of interest such as death, cure, heart attack, failure, etc. In these types of designs, we must run the experiment long enough and with enough subjects to achieve this many events.

You may enter a single value, a series of values, or a list of values.

The expected number of subjects, N, is given by

$$N = \frac{E}{1 - C/100}$$

where C is the percent censored.

This calculation assumes that the experiment is run until E events occur.

Percent Censored

This is the percent of the subjects (items or objects) that are censored. These are those that have not exhibited the event of interest by the end of the study.

Only one value may be entered.

Precision

Width of Confidence Interval (Two-Sided)

The width is the distance between the lower confidence limit (LCL) and the upper confidence limit (UCL), calculated as UCL - LCL. The unit of measure is that of the reliability, R(t). Since R(t) is a probability that ranges between 0 and 1, this value, which is a difference between two such values, also ranges between 0 and 1.

You can enter a single value, a list of values, or a series of values.

Distance from tp to Limit (One-Sided)

This precision measure is the distance from the reliability R(t) to the lower confidence limit (LCL) or the upper confidence limit (UCL). It is calculated using either |R(t)-LCL| or |UCL-R(t)|. The unit of measure is that of the reliability, R(t). Since R(t) is a probability that ranges between 0 and 1, this value, which is a difference between two such values, also ranges between 0 and 1.

You can enter a single value, a list of values, or a series of values.

Planning Estimate of Mean Lifetime

θ (Mean Lifetime)

θ is the Mean Lifetime. It is also called the mean failure time. You should enter the value that you would expect from a sample of this size. Note that θ is estimated by $\bar{t}$.

It is important that the value entered here is close to the actual value that will be obtained. This is accomplished using a small pilot study or prior experience with similar subjects.

θ can be any value greater than 0. You can enter a single value, a list of values, or a series of values.

Lifetimes at which Reliabilities are Computed

t (Lifetimes)

Enter one or more values for the time points at which the reliabilities are calculated. If we represent the reliability with R(t), these are the values of t. Note that reliability gradually decreases to zero as t increases. The mean time θ gives the relative scale for these time values.

The range is any value greater than zero. If the value is a long ways from the mean time, the reliability will be zero.

You may enter a single value, a list of values, or a series of values.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for R(t) such that the width of the interval is 0.05 when t is 1, 2, or 3. The percent censored is anticipated to be about 20% and the mean lifetime is 1.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for Exponential Reliability** procedure window by expanding **Survival,** then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for Exponential Reliability**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	**Value**
Design Tab
Solve For	**Number of Events**
Interval Type	**Two-Sided**
Confidence Level (1 – Alpha)	**0.95**
Percent Censored..................................	**20**
Width of Confidence Interval	**0.05**
θ (Mean Lifetime)...................................	**1**
t (Lifetimes)	**1 2 3**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Two-Sided Confidence Intervals for Exponential Reliability

Conf Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Mean Lifetime θ	Time t	Reliability at Time t R(t)	C.I. Lower Limit	C.I. Upper Limit
0.950	208	260	0.1000	0.0998	1.000	1.00	0.3679	0.3197	0.4195
0.950	114	143	0.1000	0.0999	1.000	2.00	0.1353	0.0922	0.1921
0.950	40	50	0.1000	0.0989	1.000	3.00	0.0498	0.0183	0.1173

Report Definitions
Confidence Level is the proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of R(t).
Number of Events E is the number of events that must occur before the experiment can be stopped.
Expected Sample Size N is the anticipated number of subjects that must be sampled so that the desired number of events occur.
Target Width is the width that was requested.
Actual Width is the calculated width. This is slightly different from the Target Width because E is an integer.
Mean Lifetime θ is the total lifetime of all subjects divided by E, the number of events.
Time t is the time at which the reliability is calculated.
Reliability R(t) is the probability of surviving to time t.
C.I. Lower Limit and C.I. Upper Limit is the confidence interval of R(t).

## Confidence Intervals for Exponential Reliability

References
Nelson, Wayne. 1982. Applied Life Data Analysis. John Wiley & Sons. New York, New York.
Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar
    and Bailey, Inc.

Summary Statements
A total of 208 events produces a two-sided 95% confidence interval for the reliability at an
elapsed time of 1.00 with a width equal to 0.0998 when the estimate of θ is 1.000. The percent
censoring is anticipated to be 20%. These results assume Type-II Censoring in which the
experiment is run until 208 failures occur.

This report shows the calculated sample size for each of the scenarios.

Plots Section
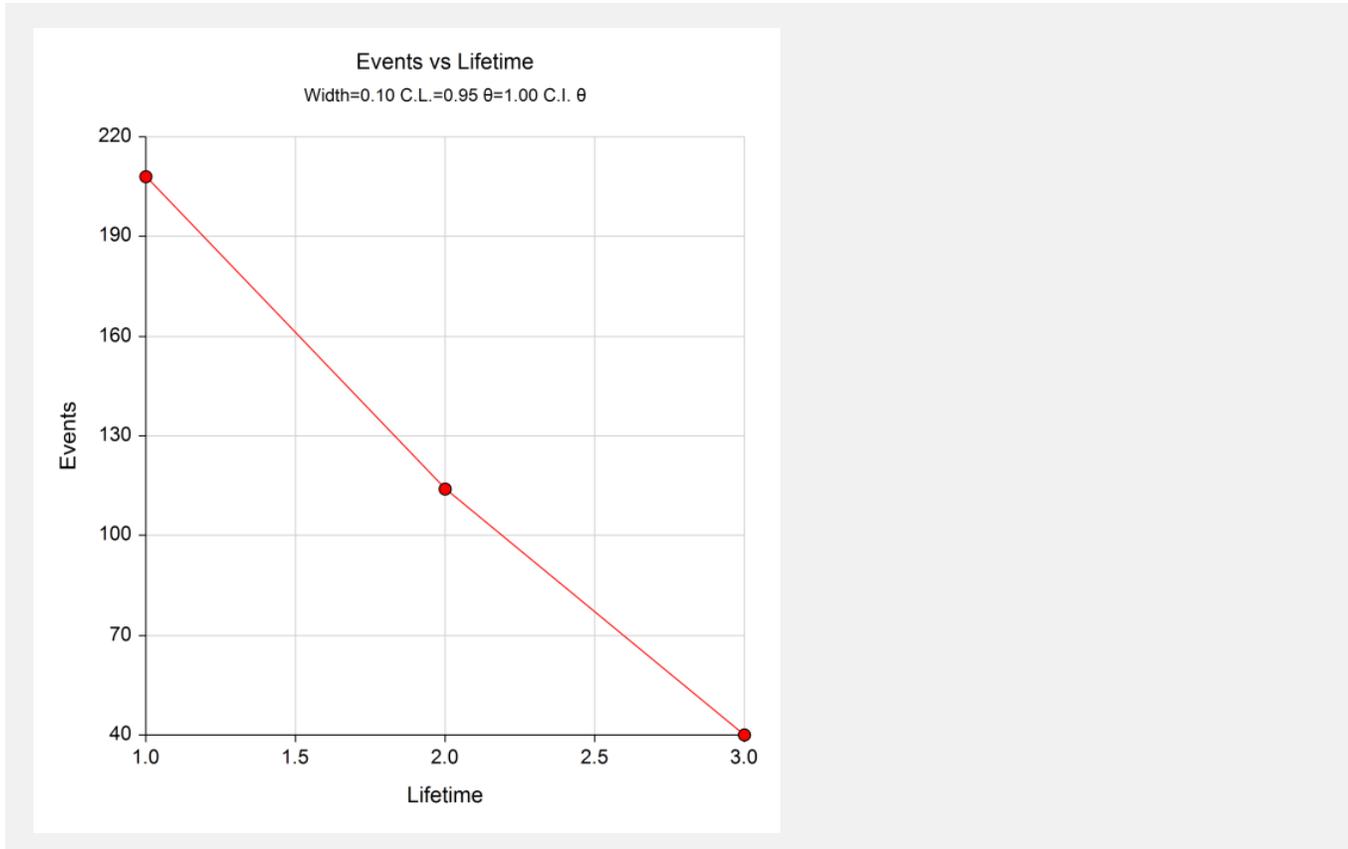
This plot shows the number of events for various time points.

Example 2 – Validation using Nelson

Nelson (1982), page 255, gives an example of that we will use to validate this procedure. In this example, the value of θ is 6097.3, the time is 60, the confidence level is 90%, and the number of events is 10. The resulting width is 0.986. Note that Nelson rounded his calculations and obtain 0.985. The reliability is found on the same page to be 0.990. Thus, the distance is 0.990 – 0.986 = 0.004.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for Exponential Reliability** procedure window by expanding **Survival,** then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for Exponential Reliability**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option   **Value**

Design Tab
Solve For .... **Number of Events**
Interval Type .... **Two-Sided**
Confidence Level (1 – Alpha) **0.95**
Number of Events **10**
Percent Censored **0**
θ (Mean Lifetime) **6097.3**
t (Lifetimes) **60**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Conf Level	Number of Events E	Expected Sample Size N	Distance from R(t) to Lower Limit	Mean Lifetime θ	Time t	Reliability at Time t R(t)	C.I. Lower Limit
0.900	10	10	0.0041	6097.300	60.00	0.9902	0.9861

PASS also calculates the distance to be 0.004.