

## Chapter 407

# Confidence Intervals for an Exponential Lifetime Percentile

## Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for a percentile of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

## Technical Details

This procedure is based on the results of Mathews (2010) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta} \exp(-t/\theta), \quad t \geq 0$$

where  $\theta$  is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate,  $\lambda$  which is equal to  $1/\theta$ . In this case, the density is

$$f(t) = \lambda \exp(-\lambda t), \quad t \geq 0.$$

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-t/\theta), \quad t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of  $\theta$  based on observing  $E$  failures in  $N$  items tested is

$$\bar{t} = \frac{1}{E} \sum_{k=1}^N t_k$$

where  $t_k$  is the amount of time that the  $k^{\text{th}}$  item was under test, whether the event of interest was observed in that item or not.

## Confidence Intervals for an Exponential Lifetime Percentile

An exact  $100(1 - \alpha)$  % confidence interval for the  $100p^{\text{th}}$  percentile of lifetime,  $T_p$ , is given by

$$P\left(\frac{2Et_p}{\chi_{1-\alpha/2}^2} \leq T_p \leq \frac{2Et_p}{\chi_{\alpha/2}^2}\right) = 1 - \alpha$$

where  $\chi_{\phi}^2$  is the value of the chi-square random variate with  $2E$  degrees of freedom that has probability  $\phi$  to the left. The value of  $t_p$  is calculate using

$$t_p = \bar{t} \ln(1 - p)$$

One-sided limits may be obtained by replacing  $\alpha/2$  by  $\alpha$ .

## Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$\text{Width} = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find  $E$ ,  $\alpha$ , or the width.

## Confidence Level

The confidence level,  $1 - \alpha$ , has the following interpretation. If thousands of samples of  $n$  items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is  $1 - \alpha$ .

## Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

## Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

### Solve For

This option specifies the parameter to be solved for from the other parameters.

### One-Sided or Two-Sided Interval

#### Interval Type

Specify whether the confidence interval for the population correlation is two-sided or one-sided. A one-sided interval is often called a **confidence bound** rather than a confidence interval because it only has one limit.

#### Two-Sided

The two-sided confidence interval is defined by two limits: an upper confidence limit (UCL) and a lower confidence limit (LCL).

These limits are constructed so that the designated proportion (confidence level) of such intervals will include the true population value.

## Confidence Intervals for an Exponential Lifetime Percentile

### Upper One-Sided

The upper confidence interval (or bound) is defined by a limit above the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value below it.

### Lower One-Sided

The lower confidence interval (or bound) is defined by a limit below the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value above it.

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## Confidence

### Confidence Level

The confidence level,  $1 - \alpha$ , has the following interpretation. If thousands of samples with  $E$  failures are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is  $1 - \alpha$ .

Often, the values 0.95 or 0.99 are used. You can enter single values or a range of values such as *0.90, 0.95* or *0.90 to 0.99 by 0.01*.

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## Number of Events and Censoring

### E (Number of Events)

Enter a value for the number of events,  $E$ . This is the number of subjects that exhibit the event of interest such as death, cure, heart attack, failure, etc. In these types of designs, we must run the experiment long enough and with enough subjects to achieve this many events.

You may enter a single value, a series of values, or a list of values.

The expected number of subjects,  $N$ , is given by

$$N = \frac{E}{1 - C/100}$$

where  $C$  is the percent censored.

This calculation assumes that the experiment is run until  $E$  events occur.

### Percent Censored

This is the percent of the subjects (items or objects) that are censored. These are those that have not exhibited the event of interest by the end of the study.

Only one value may be entered.

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## Precision

### Width of Confidence Interval (Two-Sided)

The width is the distance between the lower confidence limit (LCL) and the upper confidence limit (UCL), calculated as  $UCL - LCL$ . The unit of measure is that of the Mean Lifetime or  $\theta$ , which is usually time (hours, days, months, or years).

A reasonable width may be determined as a percentage of the mean lifetime. For example, suppose  $\theta$  is 20 months. You might want a sample size large enough so that you can estimate  $\theta$  within plus or minus 5%, or a width of 10%. Since 10% of  $\theta$  is 2 months, the width you would enter is '2.'

This width can be any value greater than zero. You can enter a single value, a list of values, or a series of values.

## Confidence Intervals for an Exponential Lifetime Percentile

This measure is scaled in the units of the mean survival time,  $\theta$ . If you want to use **relative width**, set  $\theta$  to '1' and enter the relative width as a proportion here. For example, for 20%, you would enter '0.2' here.

### Distance from $t_p$ to Limit (One-Sided)

This precision measure is the distance from  $t_p$  to the lower confidence limit (*LCL*) or the upper confidence limit (*UCL*). It is calculated using  $|t_p - LCL|$  or  $|UCL - t_p|$ . The unit of measure is that of the mean lifetime,  $\theta$ , which is usually time (hours, days, months, or years).

You can enter any value greater than zero. You can enter a single value, a list of values, or a series of values.

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## Planning Estimate of Mean Lifetime

### $\theta$ (Mean Lifetime)

$\theta$  is the Mean Lifetime. It is also called the mean failure time. You should enter the value that you would expect from a sample of this size. Note that  $\theta$  is estimated by  $\bar{t}$ .

It is important that the value entered here is close to the actual value that will be obtained. This is accomplished using a small pilot study or prior experience with similar subjects.

$\theta$  can be any value greater than 0. You can enter a single value, a list of values, or a series of values.

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## Exponential Lifetime-Percentile Percentages

### 100p (Percentile Percentages)

Enter one or more values for the lifetime-percentile percentage, 100p. The lifetime percentile,  $t_p$ , is defined as the specific time by which 100p% of the items are expected to have failed.

So, entering 20 here would indicate that you want the width or number of events for the 20th percentile: the time at which 20% of the items (subjects, units, etc.) are expected to have failed.

Entering '50' here will cause the analysis of the confidence interval for the median lifetime.

Entering '63.2' here will cause the analysis of the confidence interval for the mean lifetime.

The range is  $0 < \text{Percentage} < 100$ . Note that you cannot enter 0 or 100.

You may enter a single value, a list of values, or a series of values.

## Confidence Intervals for an Exponential Lifetime Percentile

## Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for  $T_p$  such that the width of the interval is 0.10 when 100p is 20%, 30%, or 40%. The percent censored is anticipated to be about 20% and the mean lifetime is 1.

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for an Exponential Lifetime Percentile** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for an Exponential Lifetime Percentile**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	Number of Events
Interval Type .....	Two-Sided
Confidence Level (1 – Alpha) .....	0.95
Percent Censored.....	20
Width of Confidence Interval .....	0.1
$\theta$ (Mean Lifetime).....	1
100p (Percentile Percentages) .....	20 30 40

### Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

#### Numeric Results

##### Numeric Results for Two-Sided Confidence Intervals for Exponential Lifetime Percentile

Conf Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Mean Lifetime $\theta$	Percentile Percentage 100p	Lifetime Percentile tp	C.I. Lower Limit	C.I. Upper Limit
0.950	81	102	0.100	0.099	1.000	20.0	0.223	0.182	0.281
0.950	200	250	0.100	0.100	1.000	30.0	0.357	0.312	0.412
0.950	405	507	0.100	0.100	1.000	40.0	0.511	0.465	0.564

##### Report Definitions

Confidence Level is the proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of  $t_p$ .

Number of Events E is the number of events that must occur before the experiment can be stopped.

Expected Sample Size N is the anticipated number of subjects that must be sampled so that the desired number of events occur.

Target Width is the width that was requested.

Actual Width is the calculated width. This is slightly different from the Target Width because E is an integer.

Mean Lifetime  $\theta$  is the total lifetime of all subjects divided by E, the number of events.

Percentile Percentage 100p is the percentage of the percentile.

Lifetime Percentile  $t_p$  is the 100pth percentile of these exponentially distributed lifetimes.

C.I. Lower Limit and C.I. Upper Limit is the confidence interval of  $t_p$ .

##### References

Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

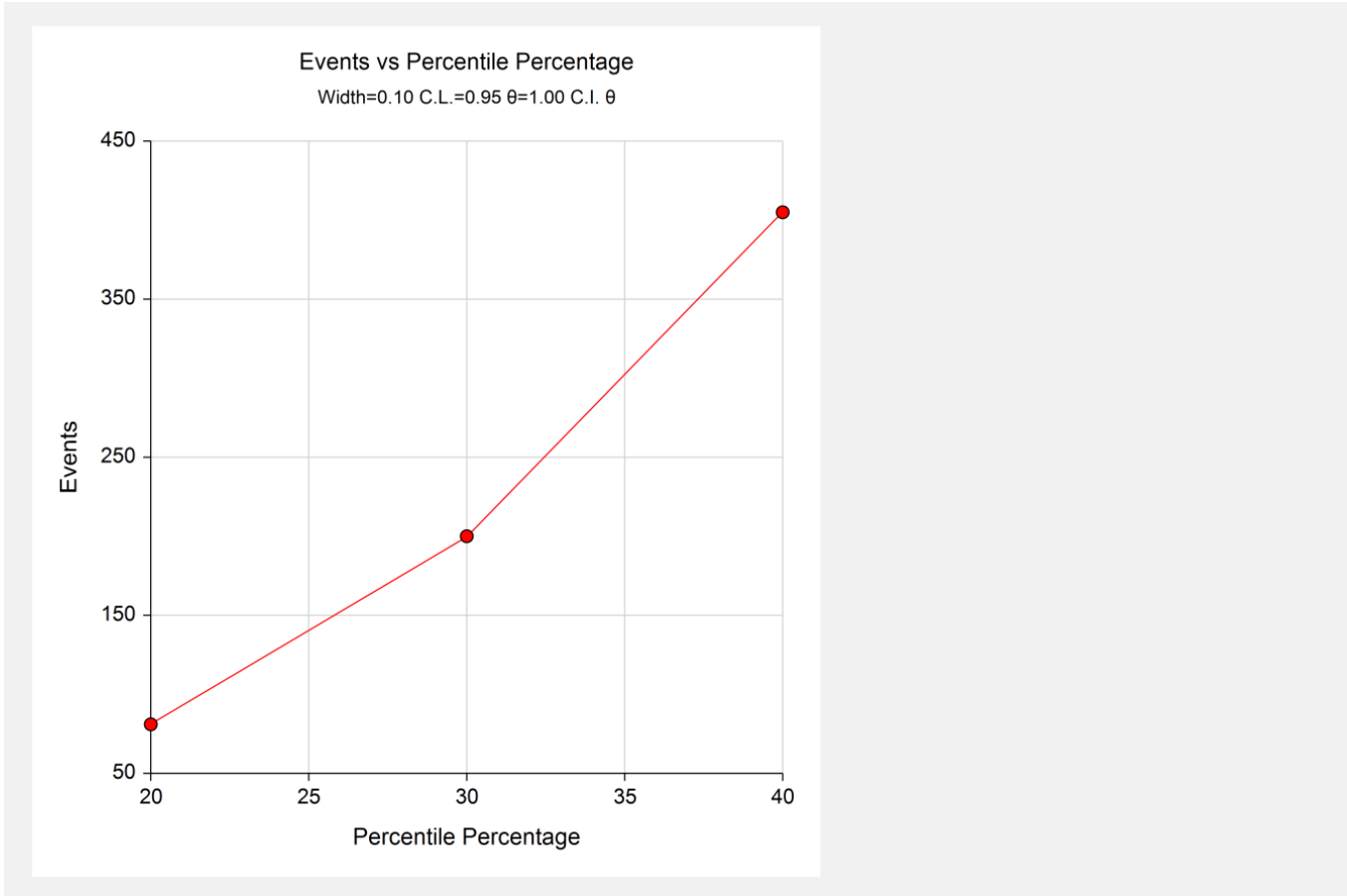
## Confidence Intervals for an Exponential Lifetime Percentile

### Summary Statements

A total of 81 events produces a two-sided 95% confidence interval with a width equal to 0.099 when the estimate of  $\theta$  is 1.000. The percent censoring is anticipated to be 20%. These results assume Type-II Censoring in which the experiment is run until 81 failures occur.

This report shows the calculated sample size for each of the scenarios.

### Plots Section



This plot shows the number of events for various percentile percentages.

## Confidence Intervals for an Exponential Lifetime Percentile

## Example 2 – Validation using Mathews

Mathews (2010), page 193, gives an example of a sample size calculation. In this example the value of  $\theta$  is 1.0, the percentile percentage is 63.2 (the mean), the confidence level is 95%, and the width is 0.40 (plus or minus 0.2). The resulting number of events is 97. Note that Mathews uses a normal approximation to the chi-square distribution which makes his results a little different than ours.

### Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for an Exponential Lifetime Percentile** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for an Exponential Lifetime Percentile**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
<b>Design Tab</b>	
Solve For .....	<b>Number of Events</b>
Interval Type .....	<b>Two-Sided</b>
Confidence Level (1 – Alpha) .....	<b>0.95</b>
Percent Censored.....	<b>20</b>
Width of Confidence Interval .....	<b>0.1</b>
$\theta$ (Mean Lifetime).....	<b>1</b>
100p (Percentile Percentages) .....	<b>63.2</b>

### Output

Click the Calculate button to perform the calculations and generate the following output.

#### Numeric Results

Conf Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Mean Lifetime $\theta$	Percentile Percentage 100p	Lifetime Percentile tp	C.I. Lower Limit	C.I. Upper Limit
0.950	100	125	0.400	0.399	1.000	63.2	1.000	0.829	1.229

PASS calculates the number of events to be 100. The difference between 100 and the 97 that Mathews obtained is because Mathews used a normal approximation to the chi-square distribution. PASS used exact chi-square results.