Chapter 472

Confidence Intervals for the Difference Between Two Means with Tolerance Probability

Introduction

This procedure calculates the sample size necessary to achieve a specified distance from the difference in sample means to the confidence limit(s) with a given tolerance probability at a stated confidence level for a confidence interval about the difference in means when the underlying data distribution is normal.

Sample sizes are calculated only for the case where the standard deviations are assumed to be equal, wherein the pooled standard deviation formula is used.

Technical Details

Let the means of the two populations be represented by μ_1 and μ_2, and let the standard deviations of the two populations be represented as σ_1 and σ_2.

When $\sigma_1 = \sigma_2 = \sigma$ are unknown, the appropriate two-sided confidence interval for $\mu_1 - \mu_2$ is

$$\overline{X}_1 - \overline{X}_2 \pm t_{1-\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Upper and lower one-sided confidence intervals can be obtained by replacing $\alpha/2$ with α.

The required sample size for a given precision, D, can be found by solving the following equation iteratively

$$D = t_{1-\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

This equation can be used to solve for D or n_1 or n_2 based on the values of the remaining parameters.

There is an additional subtlety that arises when the standard deviation is to be chosen for estimating sample size. The sample sizes determined from the formula above produce confidence intervals with the specified widths only when the future samples have a pooled standard deviation that is no greater than the value specified.

As an example, suppose that 15 individuals are sampled from each population in a pilot study, and a pooled standard deviation estimate of 5.4 is obtained from the sample. The purpose of a later study is to estimate the difference in means within 10 units. Suppose further that the sample size needed is calculated to be 62 per group using the formula above with 5.4 as the estimate for the pooled standard deviation. The samples of size 62 are then obtained from each population, but the pooled standard deviation turns out to be 6.3 rather than 5.4. The confidence interval is computed and the distance from the difference in means to the confidence limits is greater than 10 units.

This example illustrates the need for an adjustment to adjust the sample size such that the distance from the difference in means to the confidence limits will be below the specified value with known probability.

Such an adjustment for situations where a previous sample is used to estimate the standard deviation is derived by Harris, Horvitz, and Mood (1948) and discussed in Zar (1984). The adjustment is

$$D = t_{1-\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \sqrt{F_{1-\gamma; n_1+n_2-2, m_1+m_2-2}}$$

where $1 - \gamma$ is the probability that the distance from the difference in means to the confidence limit(s) will be below the specified value, and m_1 and m_2 are the sample sizes in the previous samples that were used to estimate the pooled standard deviation.

The corresponding adjustment when no previous sample is available is discussed in Kupper and Hafner (1989). The adjustment in this case is

$$D = t_{1-\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \sqrt{\frac{\chi^2_{1-\gamma, n_1+n_2-2}}{n_1 + n_2 - 2}}$$

where, again, $1 - \gamma$ is the probability that the distance from the difference in means to the confidence limit(s) will be below the specified value.

Each of these adjustments accounts for the variability in a future estimate of the pooled standard deviation. In the first adjustment formula (Harris, Horvitz, and Mood, 1948), the distribution of the pooled standard deviation is based on the estimate from previous samples. In the second adjustment formula, the distribution of the pooled standard deviation is based on a specified value that is assumed to be the population pooled standard deviation.

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of n_1 and n_2 items are drawn from populations using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population mean difference is $1 - \alpha$.

Notice that is a long term statement about many, many samples.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters.

One-Sided or Two-Sided Interval

Interval Type

Specify whether the interval to be used will be a one-sided or a two-sided confidence interval.

Confidence and Tolerance

Confidence Level (1 – Alpha)

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of n_1 and n_2 items are drawn from populations using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population mean difference is $1 - \alpha$.

Often, the values 0.95 or 0.99 are used. You can enter single values or a range of values such as *0.90 ,0.95* or *0.90 to 0.99 by 0.01*.

Tolerance Probability

This is the probability that a future interval with sample sizes N1 and N2 and the specified confidence level will have a distance from the difference in means to the limit(s) that is less than or equal to the distance specified.

If a tolerance probability is not used, as in the 'Confidence Intervals for the Difference between Two Means' procedure, the sample size is calculated for the expected distance from the difference in means to the limit(s), which assumes that the future standard deviation will also be the one specified.

Using a tolerance probability implies that the standard deviation of the future sample will not be known in advance, and therefore, an adjustment is made to the sample size formula to account for the variability in the standard deviation. Use of a tolerance probability is similar to using an upper bound for the standard deviation in the 'Confidence Intervals for the Difference between Two Means' procedure.

The range of values that can be entered here is values between 0 and 1.

You can enter a range of values such as *.70 .80 .90* or *.70 to .95 by .05*.

Sample Size (When Solving for Sample Size)

Group Allocation

Select the option that describes the constraints on *N1* or *N2* or both.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. Since you are solving for both sample sizes at once, no additional sample size parameters need to be entered.

- **Enter N1, solve for N2**

  Select this option when you wish to fix *N1* at some value (or values), and then solve only for *N2*. Please note that for some values of *N1*, there may not be a value of *N2* that is large enough to obtain the desired power.

- **Enter N2, solve for N1**

  Select this option when you wish to fix *N2* at some value (or values), and then solve only for *N1*. Please note that for some values of *N2*, there may not be a value of *N1* that is large enough to obtain the desired power.

- **Enter R = N2/N1, solve for N1 and N2**

  For this choice, you set a value for the ratio of *N2* to *N1*, and then PASS determines the needed *N1* and *N2*, with this ratio, to obtain the desired power. An equivalent representation of the ratio, *R*, is

  $$N2 = R * N1.$$

- **Enter percentage in Group 1, solve for N1 and N2**

  For this choice, you set a value for the percentage of the total sample size that is in Group 1, and then PASS determines the needed *N1* and *N2* with this percentage to obtain the desired power.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1, solve for N2"

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed if Group Allocation = "Enter N2, solve for N1"

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter R = N2/N1, solve for N1 and N2."

R is the ratio of *N2* to *N1*. That is,

$$R = N2 / N1.$$

Use this value to fix the ratio of *N2* to *N1* while solving for *N1* and *N2*. Only sample size combinations with this ratio are considered.

N2 is related to *N1* by the formula:

$$N2 = [R \times N1],$$

where the value *[Y]* is the next integer $\geq Y$.

For example, setting $R = 2.0$ results in a Group 2 sample size that is double the sample size in Group 1 (e.g., $N1 = 10$ and $N2 = 20$, or $N1 = 50$ and $N2 = 100$).

R must be greater than 0. If $R < 1$, then $N2$ will be less than $N1$; if $R > 1$, then $N2$ will be greater than $N1$. You can enter a single or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter percentage in Group 1, solve for N1 and N2."

Use this value to fix the percentage of the total sample size allocated to Group 1 while solving for $N1$ and $N2$. Only sample size combinations with this Group 1 percentage are considered. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single or a series of values.

Sample Size (When <u>Not</u> Solving for Sample Size)

Group Allocation

Select the option that describes how individuals in the study will be allocated to Group 1 and to Group 2.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. A single per group sample size will be entered.

- **Enter N1 and N2 individually**

  This choice permits you to enter different values for $N1$ and $N2$.

- **Enter N1 and R, where N2 = R * N1**

  Choose this option to specify a value (or values) for $N1$, and obtain $N2$ as a ratio (multiple) of $N1$.

- **Enter total sample size and percentage in Group 1**

  Choose this option to specify a value (or values) for the total sample size (N), obtain $N1$ as a percentage of N, and then $N2$ as $N - N1$.

Sample Size Per Group

This option is displayed only if Group Allocation = "Equal (N1 = N2)."

The Sample Size Per Group is the number of items or individuals sampled from each of the Group 1 and Group 2 populations. Since the sample sizes are the same in each group, this value is the value for $N1$, and also the value for $N2$.

The Sample Size Per Group must be ≥ 2. You can enter a single value or a series of values.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1 and N2 individually" or "Enter N1 and R, where N2 = R * N1."

$N1$ is the number of items or individuals sampled from the Group 1 population.

$N1$ must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed only if Group Allocation = "Enter N1 and N2 individually."

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter N1 and R, where N2 = R * N1."

R is the ratio of *N2* to *N1*. That is,

$$R = N2/N1$$

Use this value to obtain *N2* as a multiple (or proportion) of *N1*.

N2 is calculated from *N1* using the formula:

$$N2=[R \ x \ N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1.

R must be greater than 0. If *R* < 1, then *N2* will be less than N1; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single value or a series of values.

Total Sample Size (N)

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This is the total sample size, or the sum of the two group sample sizes. This value, along with the percentage of the total sample size in Group 1, implicitly defines *N1* and *N2*.

The total sample size must be greater than one, but practically, must be greater than 3, since each group sample size needs to be at least 2.

You can enter a single value or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This value fixes the percentage of the total sample size allocated to Group 1. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single value or a series of values.

Precision

Distance from Mean Difference to Limit(s)

This is the distance from the confidence limit(s) to the difference in means. For two-sided intervals, it is also known as the precision, half-width, or margin of error.

You can enter a single value or a list of values. The value(s) must be greater than zero.

Pooled Standard Deviation

Standard Deviation Source

This procedure permits two sources for estimates of the pooled standard deviation:

- **S is a Population Standard Deviation**

  This option should be selected if there are no previous samples that can be used to obtain an estimate of the pooled standard deviation. In this case, the algorithm assumes that the future sample obtained will be from a population with standard deviation S.

- **S from a Previous Sample**

  This option should be selected if the estimate of the pooled standard deviation is obtained from previous random samples from the same distributions as those to be sampled. The total sample size of the previous samples must also be entered under 'Total Sample Size of Previous Sample'.

Pooled Standard Deviation – Population Standard Deviation

S (Standard Deviation)

Enter an estimate of the pooled standard deviation (must be positive). In this case, the algorithm assumes that future samples obtained will be from a population with pooled standard deviation S.

One common method for estimating the standard deviation is the range divided by 4, 5, or 6.

You can enter a range of values such as *1 2 3* or *1 to 10 by 1*.

Press the Standard Deviation Estimator button to load the Standard Deviation Estimator window.

Pooled Standard Deviation – Standard Deviation from Previous Sample

S (Standard Deviation)

Enter an estimate of the pooled standard deviation from a previous (or pilot) study. This value must be positive.

A range of values may be entered.

Press the Standard Deviation Estimator button to load the Standard Deviation Estimator window.

Total Sample Size of Previous Sample

Enter the total sample size that was used to estimate the pooled standard deviation entered in S (SD Estimated from a Previous Sample). The total sample size should be the total of the two sample sizes ($m_1 + m_2$) that were used to estimate the pooled standard deviation.

If the previous sample used for the estimate of the pooled standard deviation is a single sample rather than two samples, enter the sample size of the previous sample plus one.

This value is entered only when 'Standard Deviation Source:' is set to 'S from a Previous Sample'.

Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for the difference between two population means. It is very important that the mean weight is estimated within 10 units. The pooled standard deviation estimate, based on the range of data values, is 25.6. Instead of examining only the interval half-width of 10, a series of half-widths from 5 to 15 will also be considered.

The goal is to determine the sample size necessary to obtain a two-sided confidence interval such that the difference in means is estimated within 10 units. Tolerance probabilities of 0.70 to 0.95 will be examined.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Difference Between Two Means with Tolerance Probability** procedure window by expanding **Means**, then **Two Independent Means**, then clicking on **Confidence Interval**, and then clicking on **Confidence Intervals for the Difference Between Two Means with Tolerance Probability**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option   **Value**

Design Tab
Solve For .... **Sample Size**
Interval Type .... **Two-Sided**
Confidence Level **0.95**
Tolerance Probability .... **0.70 to 0.95 by 0.05**
Group Allocation .... **Equal (N1 = N2)**
Distance from Mean Diff to Limit(s) **10**
Standard Deviation Source..................... **S is a Population Standard Deviation**
S... **25.6**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Two-Sided Confidence Intervals for the Difference in Means

Confidence Level	N1	N2	N	Target Dist from Mean Diff to Limits	Actual Dist from Mean Diff to Limits	Pooled Standard Deviation	Tolerance Probability
0.950	55	55	110	10.000	9.994	25.60	0.700
0.950	56	56	112	10.000	9.998	25.60	0.750
0.950	58	58	116	10.000	9.919	25.60	0.800
0.950	59	59	118	10.000	9.951	25.60	0.850
0.950	61	61	122	10.000	9.921	25.60	0.900
0.950	63	63	126	10.000	9.962	25.60	0.950

References
Kupper, L. L. and Hafner, K. B. 1989. 'How Appropriate are Popular Sample Size Formulas?', The American
    Statistician, Volume 43, No. 2, pp. 101-105.

Confidence Intervals for the Difference Between Two Means with Tolerance Probability

Report Definitions

Confidence level is the proportion of confidence intervals (constructed with this same confidence level,
   sample size, etc.) that would contain the true difference in population means.

N1 and N2 are the number of items sampled from each population.

N is the total sample size, N1 + N2.

Target Dist from Mean Diff to Limit is the value of the distance that is entered into the procedure.

Actual Dist from Mean Diff to Limit is the value of the distance that is obtained from the procedure.

Pooled Standard Deviation is the standard deviation upon which the distance from mean difference to limit
   calculations are based.

Tolerance Probability is the probability that a future interval with sample size N and corresponding
   confidence level will have a distance from the mean to the limit(s) that is less than or equal to the
   specified distance.

Summary Statements

The probability is 0.700 that group sample sizes of 55 and 55 will produce a two-sided 95%
confidence interval with a distance from the difference in means to the limits that is less
than or equal to 9.994 if the pooled standard deviation is 25.60.

This report shows the calculated sample size for each of the scenarios.

Plots Section
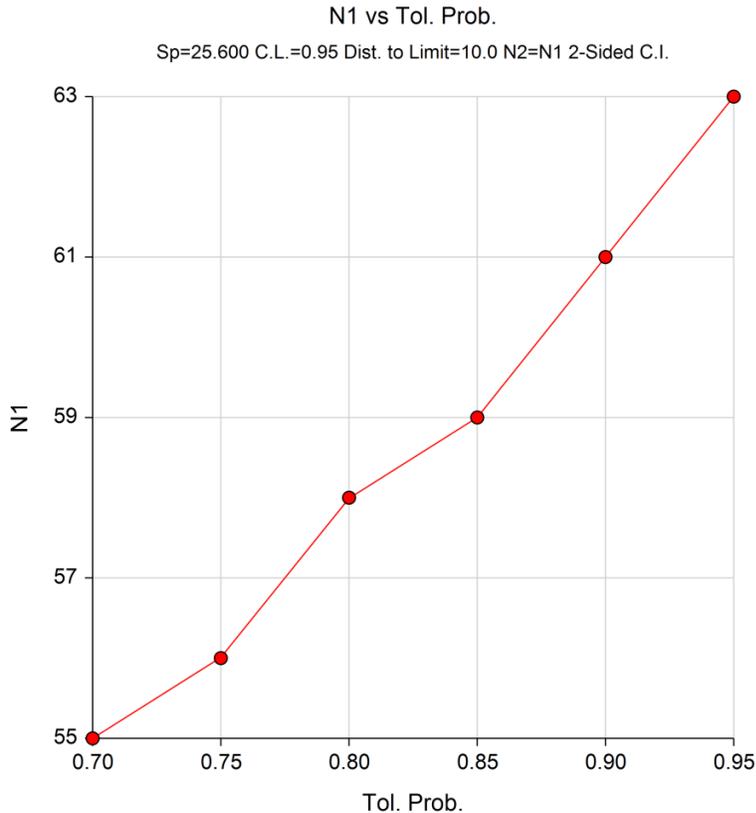
This plot shows the sample size of each group versus the precision for the two confidence levels.

Example 2 – Validation using Zar

Zar (1984) pages 133-134 gives an example of a precision calculation for a confidence interval for the difference between two means when the confidence level is 95%, the pooled standard deviation is 0.720625 from a total sample size of 13, the precision is 0.5, and the tolerance probability is 0.90. The sample size for each group is determined to be 34.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Difference Between Two Means with Tolerance Probability** procedure window by expanding **Means**, then **Two Independent Means**, then clicking on **Confidence Interval**, and then clicking on **Confidence Intervals for the Difference Between Two Means with Tolerance Probability**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option   **Value**

Design Tab

Solve For .... **Sample Size**

Interval Type .... **Two-Sided**

Confidence Level.................................... **0.95**

Tolerance Probability............................. **0.90**

Group Allocation .... **Equal (N1 = N2)**

Distance from Mean Diff to Limit(s) **0.5**

Standard Deviation Source..................... **S from a Previous Sample**

S.. **0.720625**

Total Sample Size of Previous Sample .. **13**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Two-Sided Confidence Intervals for the Difference in Means

Confidence Level	N1	N2	N	Target Dist from Mean Diff to Limits	Actual Dist from Mean Diff to Limits	Pooled Standard Deviation	Tolerance Probability
0.950	34	34	68	0.500	0.496	0.72	0.900

Total sample size for estimate of pooled standard deviation from previous samples = 13.

PASS also calculated the sample size in each group to be 34.