

Chapter 409

Confidence Intervals for the Exponential Hazard Rate

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the hazard rate of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Nelson (1982) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \lambda \exp(-\lambda t), \quad t \geq 0$$

where λ is the hazard (failure) rate.

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-\lambda t), \quad t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-\lambda t)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of λ based on observing E failures in N items tested is

$$h = \frac{E}{\sum_{k=1}^N t_k}$$

where t_k is the amount of time that the k^{th} subject was under test, whether the event of interest was observed in that subject or not.

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An exact $100(1 - \alpha)$ % confidence interval for λ is given by

$$P\left(\frac{h\chi_{\alpha/2}^2}{2E} \leq \lambda \leq \frac{h\chi_{1-\alpha/2}^2}{2E}\right) = 1 - \alpha$$

where χ_{ϕ}^2 is the value of the chi-square random variate with $2E$ degrees of freedom that has probability ϕ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α .

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find E , α , or the width.

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of n items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

This option specifies the parameter to be solved for from the other parameters.

One-Sided or Two-Sided Interval

Interval Type

Specify whether the confidence interval for the population correlation is two-sided or one-sided. A one-sided interval is often called a **confidence bound** rather than a confidence interval because it only has one limit.

Two-Sided

The two-sided confidence interval is defined by two limits: an upper confidence limit (UCL) and a lower confidence limit (LCL).

These limits are constructed so that the designated proportion (confidence level) of such intervals will include the true population value.

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Upper One-Sided

The upper confidence interval (or bound) is defined by a limit above the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value below it.

Lower One-Sided

The lower confidence interval (or bound) is defined by a limit below the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value above it.

Confidence

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples with E failures are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Often, the values 0.95 or 0.99 are used. You can enter single values or a range of values such as *0.90, 0.95* or *0.90 to 0.99 by 0.01*.

Number of Events and Censoring

E (Number of Events)

Enter a value for the number of events, E . This is the number of subjects that exhibit the event of interest such as death, cure, heart attack, failure, etc. In these types of designs, we must run the experiment long enough and with enough subjects to achieve this many events.

You may enter a single value, a series of values, or a list of values.

The expected number of subjects, N , is given by

$$N = \frac{E}{1 - C/100}$$

where C is the percent censored.

This calculation assumes that the experiment is run until E events occur.

Percent Censored

This is the percent of the subjects (items or objects) that are censored. These are those that have not exhibited the event of interest by the end of the study.

Only one value may be entered.

Precision

Width of Confidence Interval (Two-Sided)

The width is the distance between the lower confidence limit (LCL) and the upper confidence limit (UCL), calculated as $UCL - LCL$. The unit of measure is that of the hazard rate or λ , which is usually a fraction of a count such as 5.4 per year.

This width can be any value greater than zero. You can enter a single value, a list of values, or a series of values.

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Distance from λ to Limit (One-Sided)

This precision measure is the distance from λ to the lower confidence limit (LCL) or the upper confidence limit (UCL). It is calculated using $|\lambda - \text{LCL}|$ or $|\text{UCL} - \lambda|$. The unit of measure is that of the hazard rate or λ , which is usually a fraction of a count such as 5.4 per year.

You can enter any value greater than zero. You can enter a single value, a list of values, or a series of values.

Planning Estimate of Hazard Rate

λ (Mean Lifetime)

λ is the hazard rate or failure rate. Note that λ is estimated by h (see above).

The unit of measure is a fraction of a count such as 5.4 per year.

It is important that the value entered here is close to the actual value that will be obtained. This is accomplished using a small pilot study or prior experience with similar subjects.

λ can be any value greater than 0. You can enter a single value, a list of values, or a series of values.

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Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for the hazard rate such that the width of the interval is 0.4 or 0.6. The hazard rate is estimated to be between 1 and 3. The percent censored is anticipated to be about 20%.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Exponential Hazard Rate** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for the Exponential Hazard Rate**. You may then make the appropriate entries as listed below or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Number of Events
Interval Type	Two-Sided
Confidence Level (1 – Alpha)	0.95
Percent Censored.....	20
Width of Confidence Interval	0.4 0.6
λ (Hazard Rate)	1.0 1.5 2.0 2.5 3.0

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Two-Sided Confidence Intervals for Exponential Hazard Rate

Confidence Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Hazard Rate λ	C.I. Lower Limit	C.I. Upper Limit
0.950	96	120	0.400	0.400	1.0000	0.8100	1.2097
0.950	216	270	0.400	0.400	1.5000	1.3066	1.7065
0.950	384	480	0.400	0.400	2.0000	1.8049	2.2049
0.950	601	752	0.400	0.400	2.5000	2.3041	2.7038
0.950	865	1082	0.400	0.400	3.0000	2.8034	3.2032
0.950	43	54	0.600	0.597	1.0000	0.7237	1.3203
0.950	96	120	0.600	0.600	1.5000	1.2150	1.8146
0.950	171	214	0.600	0.599	2.0000	1.7115	2.3107
0.950	267	334	0.600	0.600	2.5000	2.2091	2.8086
0.950	384	480	0.600	0.600	3.0000	2.7074	3.3074

Report Definitions

Confidence Level is the proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of λ .

Number of Events E is the number of events that must occur before the experiment can be stopped.

Expected Sample Size N is the anticipated number of subjects that must be sampled so that the desired number of events occur.

Target Width is the width that was requested.

Actual Width is the calculated width. This is slightly different from the Target Width because E is an integer.

Hazard Rate λ is the number of events E divided by the total lifetime of all subjects T.

C.I. Lower Limit and C.I. Upper Limit is the confidence interval of λ .

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References

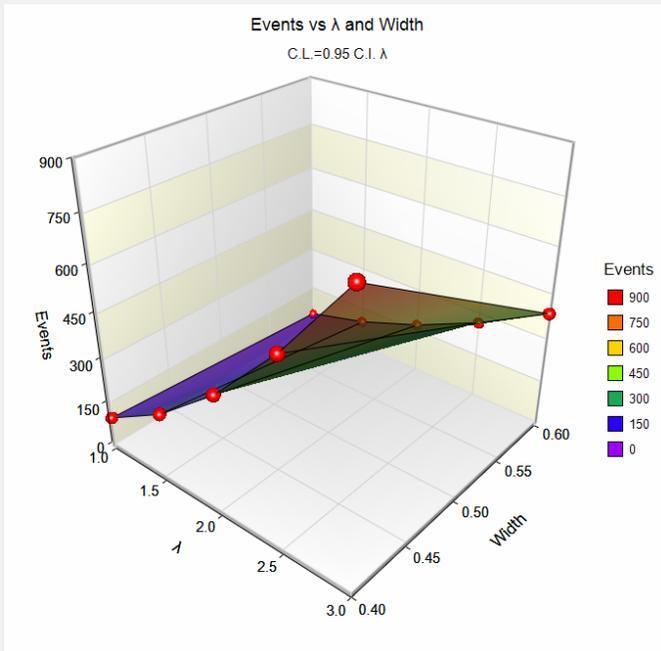
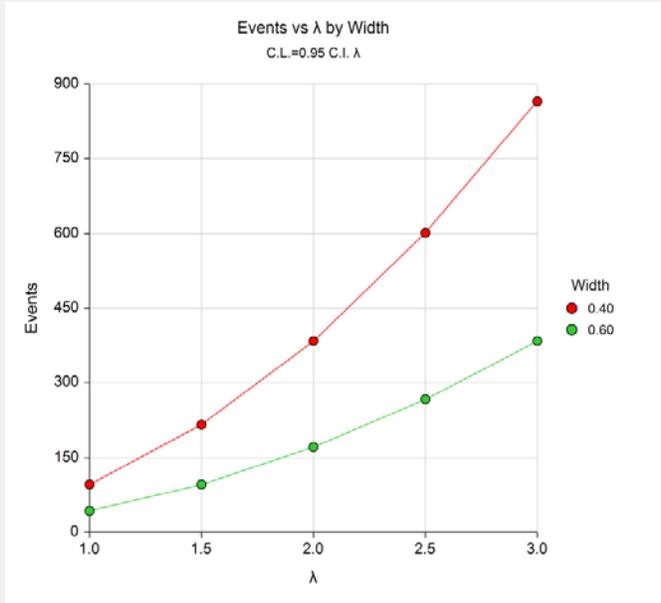
Nelson, Wayne. 1982. Applied Life Data Analysis. John Wiley & Sons. New York, New York.
 Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

Summary Statements

A total of 100 events produces a two-sided 95% confidence interval with a width equal to 0.399 when the estimate of λ (the hazard rate) is 1.0000. The percent censoring is anticipated to be 20%. These results assume Type-II Censoring in which the experiment is run until 100 failures occur.

This report shows the calculated number of events for each of the scenarios.

Plots Section



These plots show the number of events for various hazard rates.

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Example 2 – Validation using Nelson

Nelson (1982), page 254, gives an example of a calculation of a upper one-sided confidence interval for the hazard rate. In this example the value of λ is 0.000164, the confidence level is 90%, the upper confidence limit is 0.000233, and the width is 0.000069. The number of events is 10.

Note that Nelson erroneously calculated the upper limit as 1/4092.1. Using his own numbers from the bottom of page 252, the upper limit should be 1/4292.1.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Exponential Hazard Rate** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for the Exponential Hazard Rate**. You may then make the appropriate entries as listed below or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Number of Events
Interval Type	Upper One-Sided
Confidence Level (1 – Alpha)	0.90
Percent Censored.....	0
Width of Confidence Interval	0.000069
λ (Hazard Rate)	0.000164
Reports Tab	
Widths and Distances Decimals	6
λ (Hazard Rate) Decimals.....	6

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Confidence Level	Number of Events E	Expected Sample Size N	Target Distance from λ to Upper Limit	Actual Distance from λ to Upper Limit	Hazard Rate λ	C.I. Upper Limit
0.900	10	10	0.000069	0.000069	0.000164	0.000233

PASS also calculates the number of events to be 10.