

Chapter 406

Confidence Intervals for the Exponential Lifetime Mean

Introduction

This routine calculates the number of events needed to obtain a specified width of a confidence interval for the mean of an exponential distribution at a given level of confidence. The calculations assume Type-II censoring, that is, the experiment is run until a set number of events occur.

Technical Details

This procedure is based on the results of Mathews (2010) and Lawless (2003). The exponential lifetime model is based on the exponential density function

$$f(t) = \frac{1}{\theta} \exp(-t/\theta), \quad t \geq 0$$

where θ is the mean lifetime, mean failure time, mean time to failure, or mean time between failures. This model is also parameterized in terms of failure rate, λ which is equal to $1/\theta$. In this case, the density is

$$f(t) = \lambda \exp(-\lambda t), \quad t \geq 0.$$

The cumulative exponential distribution is

$$F(t) = 1 - \exp(-t/\theta), \quad t \geq 0.$$

The survival or reliability function is

$$R(t) = 1 - F(t)$$

which in the case of the exponential distribution results in

$$R(t) = \exp(-t/\theta)$$

With the assumption of Type II censoring, the maximum-likelihood estimate of θ based on observing E failures in N items tested is

$$\bar{t} = \frac{1}{E} \sum_{k=1}^N t_k$$

where t_k is the amount of time that the k^{th} item was under test, whether the event of interest was observed in that item or not.

Confidence Intervals for the Exponential Lifetime Mean

An exact $100(1 - \alpha)$ % confidence interval for θ is given by

$$P\left(\frac{2E\bar{t}}{\chi_{1-\alpha/2}^2} \leq \theta \leq \frac{2E\bar{t}}{\chi_{\alpha/2}^2}\right) = 1 - \alpha$$

where χ_{ϕ}^2 is the value of the chi-square random variate with $2E$ degrees of freedom that has probability ϕ to the left.

One-sided limits may be obtained by replacing $\alpha/2$ by α .

Confidence Interval Width

The confidence interval width, confidence level, and number of events are related in the equation

$$Width = UCL - LCL$$

where LCL and UCL are the lower and upper confidence limits.

This equation can be used to find E , α , or the width.

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples of n items are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

This option specifies the parameter to be solved for from the other parameters.

One-Sided or Two-Sided Interval

Interval Type

Specify whether the confidence interval for the population correlation is two-sided or one-sided. A one-sided interval is often called a **confidence bound** rather than a confidence interval because it only has one limit.

Two-Sided

The two-sided confidence interval is defined by two limits: an upper confidence limit (UCL) and a lower confidence limit (LCL).

These limits are constructed so that the designated proportion (confidence level) of such intervals will include the true population value.

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Upper One-Sided

The upper confidence interval (or bound) is defined by a limit above the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value below it.

Lower One-Sided

The lower confidence interval (or bound) is defined by a limit below the estimated parameter value. The limit is constructed so that the designated proportion (confidence level) of such limits has the true population value above it.

Confidence

Confidence Level

The confidence level, $1 - \alpha$, has the following interpretation. If thousands of samples with E failures are drawn from a population using simple random sampling and a confidence interval is calculated for each sample, the proportion of those intervals that will include the true population parameter is $1 - \alpha$.

Often, the values 0.95 or 0.99 are used. You can enter single values or a range of values such as *0.90, 0.95* or *0.90 to 0.99 by 0.01*.

Number of Events and Censoring

E (Number of Events)

Enter a value for the number of events, E . This is the number of subjects that exhibit the event of interest such as death, cure, heart attack, failure, etc. In these types of designs, we must run the experiment long enough and with enough subjects to achieve this many events.

You may enter a single value, a series of values, or a list of values.

The expected number of subjects, N , is given by

$$N = \frac{E}{1 - C/100}$$

where C is the percent censored.

This calculation assumes that the experiment is run until E events occur.

Percent Censored

This is the percent of the subjects (items or objects) that are censored. These are those that have not exhibited the event of interest by the end of the study.

Only one value may be entered.

Precision

Width of Confidence Interval (Two-Sided)

The width is the distance between the lower confidence limit (LCL) and the upper confidence limit (UCL), calculated as $UCL - LCL$. The unit of measure is that of the Mean Lifetime or θ , which is usually time (hours, days, months, or years).

A reasonable width may be determined as a percentage of θ . For example, suppose θ is 20 months. You might want a sample size large enough so that you can estimate θ within plus or minus 5%, or a width of 10%. Since 10% of θ is 2 months, the width you would enter is '2.'

This width can be any value greater than zero. You can enter a single value, a list of values, or a series of values.

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This measure is scaled in the units of the mean survival time, θ . If you want to use **relative width**, set θ to '1' and enter the relative width as a proportion here. For example, for 20%, you would enter '0.2' here.

Distance from θ to Limit (One-Sided)

This precision measure is the distance from θ to the lower confidence limit (LCL) or the upper confidence limit (UCL). It is calculated using $|\theta - \text{LCL}|$ or $|\text{UCL} - \theta|$. The unit of measure is that of the Mean Lifetime, θ , which is usually time (hours, days, months, or years).

A reasonable distance may be determined as a percentage of θ . For example, suppose θ is 20 months. You might want a sample size large enough so that you can estimate θ to within 5%. Since 5% of θ is 1 month, the value you would enter is '1'.

You can enter any value greater than zero. You can enter a single value, a list of values, or a series of values.

This measure is scaled in the units of the mean survival time, θ . If you want to use **relative distance**, set θ to '1' and enter the relative distance as a proportion here. For example, for 10%, you would enter '0.1' here.

Planning Estimate of Mean Lifetime

θ (Mean Lifetime)

θ is the Mean Lifetime. It is also called the mean failure time. You should enter the value that you would expect from a sample of this size. Note that θ is estimated by \bar{t} .

The unit of measure used here must also be the unit of measure used for the width or distance. If you want to make all calculation relative, enter a '1' here and then proportions as widths or distances.

It is important that the value entered here is close to the actual value that will be obtained. This is accomplished using a small pilot study or prior experience with similar subjects.

θ can be any value greater than 0. You can enter a single value, a list of values, or a series of values.

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Example 1 – Calculating Sample Size

Suppose a study is planned in which the researcher wishes to construct a two-sided 95% confidence interval for θ such that the relative width of the interval is 0.05, 0.10, 0.15, and 0.20. The percent censored is anticipated to be about 20%.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Exponential Lifetime Mean** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for the Exponential Lifetime Mean**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Number of Events
Interval Type	Two-Sided
Confidence Level (1 – Alpha)	0.95
Percent Censored.....	20
Width of Confidence Interval	0.05 to 0.2 by 0.05
θ (Mean Lifetime).....	1

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results for Two-Sided Confidence Intervals for Exponential Lifetime Mean

Confidence Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Mean Lifetime θ	C.I. Lower Limit	C.I. Upper Limit
0.950	6151	7689	0.050	0.050	1.000	0.975	1.025
0.950	1541	1927	0.100	0.100	1.000	0.952	1.052
0.950	687	859	0.150	0.150	1.000	0.929	1.079
0.950	388	485	0.200	0.200	1.000	0.908	1.107

Report Definitions

Confidence Level is the proportion of confidence intervals (constructed with this same confidence level) that would contain the true value of θ .

Number of Events E is the number of events that must occur before the experiment can be stopped.

Expected Sample Size N is the anticipated number of subjects that must be sampled so that the desired number of events occur.

Target Width is the width that was requested.

Actual Width is the calculated width. This is slightly different from the Target Width because E is an integer.

Mean Lifetime θ (or mean failure time) is the total lifetime of all subjects divided by E, the number of events.

C.I. Lower Limit and C.I. Upper Limit is the confidence interval of θ .

References

Mathews, Paul. 2010. Sample Size Calculations: Practical Methods for Engineers and Scientists. Mathews Malnar and Bailey, Inc.

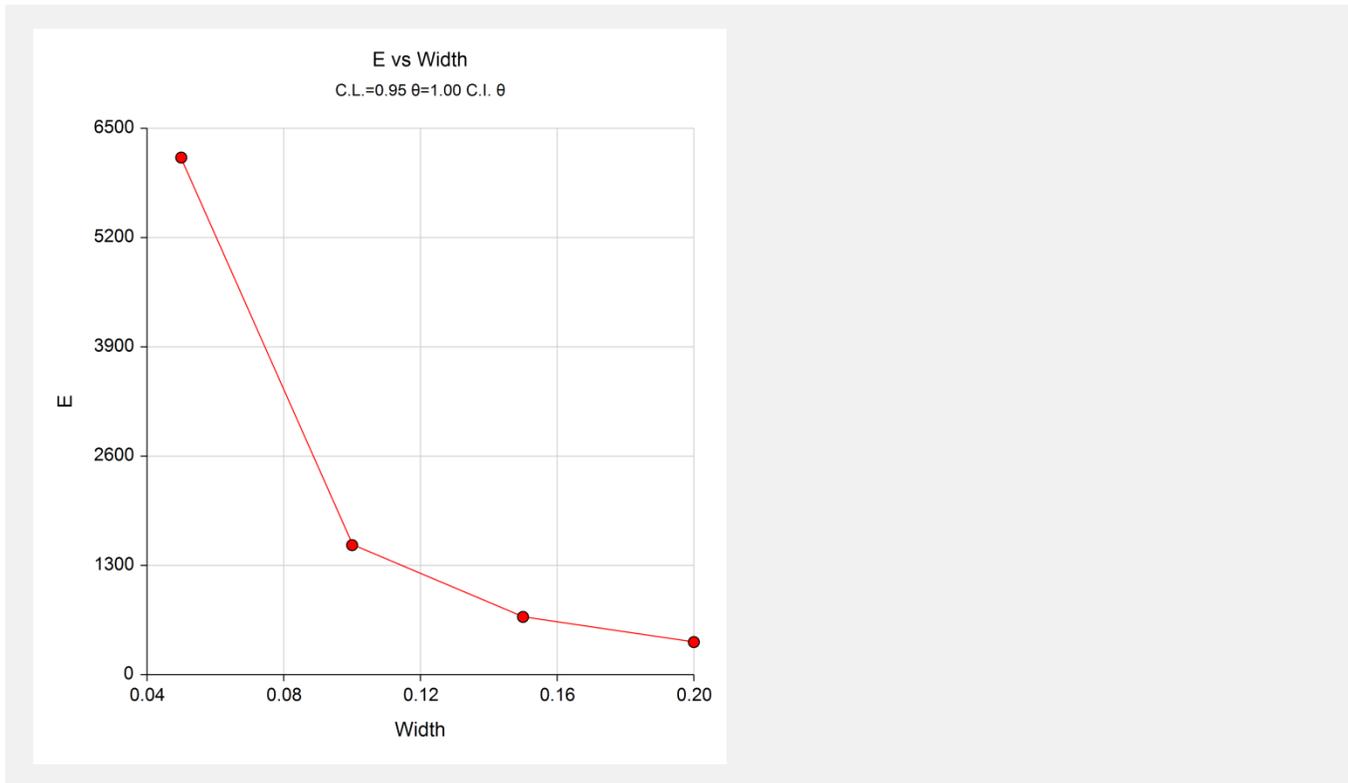
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Summary Statements

A total of 6151 events produces a two-sided 95% confidence interval with a width equal to 0.050 when the estimate of θ is 1.000. The percent censoring is anticipated to be 20%. These results assume Type-II Censoring in which the experiment is run until 6151 failures occur.

This report shows the calculated sample size for each of the scenarios.

Plots Section



This plot shows the number of events, E, for various widths.

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Example 2 – Validation using Mathews

Mathews (2010), page 192, gives an example of a sample size calculation. In this example the value of θ is 1.0, the confidence level is 95%, and the width is 0.40 (plus or minus 0.2). The resulting number of events is 97. Note that Mathews uses a normal approximation to the chi-square distribution which makes his results a little different than ours.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Confidence Intervals for the Exponential Lifetime Mean** procedure window by expanding **Survival**, then clicking on **Confidence Intervals**, and then clicking on **Confidence Intervals for the Exponential Lifetime Mean**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Number of Events
Interval Type	Two-Sided
Confidence Level (1 – Alpha)	0.95
Percent Censored.....	20
Width of Confidence Interval	0.05 to 0.2 by 0.05
θ (Mean Lifetime).....	1

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Confidence Level	Number of Events E	Expected Sample Size N	Target Width	Actual Width	Mean Lifetime θ	C.I. Lower Limit	C.I. Upper Limit
0.950	100	100	0.400	0.399	1.000	0.830	1.229

PASS calculates the number of events to be 100. The difference between 100 and the 97 that Mathews obtained is because Mathews used a normal approximation to the chi-square distribution. PASS used exact chi-square results.