Chapter 479

Non-Unity Null Tests for the Ratio of Within-Subject Variances in a Parallel Design

Introduction

This procedure calculates power and sample size of inequality tests of within-subject variabilities from a two-group, parallel design with replicates (repeated measurements) for the case when the ratio assumed by the null hypothesis is not necessarily equal to one. This routine deals with the case in which the statistical hypotheses are expressed in terms of the ratio of the within-subject variances.

Technical Details

This procedure uses the formulation given in Chow, Shao, Wang, and Lokhnygina (2018).

Suppose x_{ijk} is the response of the *i*th treatment (*i* = 1,2), *j*th subject (*j* = 1, …, *Ni*), and *k*th replicate (*k* = 1, …, M). The model analyzed in this procedure is

$$x_{ijk} = \mu_i + S_{ij} + e_{ijk}$$

where μ_i is the treatment effect, S_{ij} is the random effect of the *j*th subject in the *i*th treatment, and e_{ijk} is the within-subject error term which is normally distributed with mean 0 and variance $V_i = \sigma_{Wi}^2$.

Unbiased estimates of these variances are given by

$$\hat{V}_i = \frac{1}{N_i(M-1)} \sum_{j=1}^{N_i} \sum_{k=1}^{M} \left(x_{ijk} - \bar{x}_{ij.}\right)^2$$

A common test statistic to compare variabilities in the two groups is $T = \hat{V}_1/\hat{V}_2$. Under the usual normality assumptions, T is distributed as an F distribution with degrees of freedom $N_1(M-1)$ and $N_2(M-1)$.

Testing Variance Inequality with a Non-Unity Null

The following three sets of statistical hypotheses are used to test for variance inequality with a non-unity null

$$H_0: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} \geq R0 \text{ versus } H_1: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} < R0,$$

$$H_0: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} \leq R0 \text{ versus } H_1: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} > R0,$$

$$H_0: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} = R0 \text{ versus } H_1: \frac{\sigma_{W1}^2}{\sigma_{W2}^2} \neq R0,$$

where *R0* is the variance ratio assumed by the null hypothesis.

The corresponding test statistic is $T = (\hat{V}_1/\hat{V}_2)/R0$.

Power

The corresponding powers of these three tests are given by

$$\text{Power} = \text{P}\left(F < \frac{R0}{R1} F_{\alpha,N_1(M-1),N_2(M-1)}\right)$$

$$\text{Power} = 1 - \text{P}\left(F < \frac{R0}{R1} F_{1-\alpha,N_1(M-1),N_2(M-1)}\right)$$

$$\text{Power} = \text{P}\left(F < \frac{R0}{R1} F_{\alpha/2,N_1(M-1),N_2(M-1)}\right) + 1 - \text{P}\left(F < \frac{R0}{R1} F_{1-\alpha/2,N_1(M-1),N_2(M-1)}\right)$$

where *F* is the common F distribution with the indicated degrees of freedom, α is the significance level, and *R1* is the value of the variance ratio stated by the alternative hypothesis. Lower quantiles of F are used in the equation.

A simple binary search algorithm can be applied to this power function to obtain an estimate of the necessary sample size.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains the parameters associated with this test such as sample sizes, alpha, and power.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters. Under most situations, you will select either *Power* or *Sample Size*.

Test Direction

Alternative Hypothesis

Specify whether the alternative hypothesis of the test is one-sided or two-sided.

Note that this parameter impacts the value of alpha. The value of alpha is used directly for one-sided tests. For two-sided tests, alpha is replaced by alpha/2.

Two-Sided Hypothesis Test

H0: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} = \text{R0}$ vs. H1: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} \neq \text{R0}$

One-Sided Hypothesis Tests

Lower: H0: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} \geq \text{R0}$ vs. H1: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} < \text{R0}$

Upper: H0: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} \leq \text{R0}$ vs. H1: $\frac{\sigma_{W1}^2}{\sigma_{W2}^2} > \text{R0}$

Power and Alpha

Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis and is equal to one minus Beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected. In this procedure, a type-I error occurs when you reject the null hypothesis when it is true.

Values must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01*.

Sample Size (When Solving for Sample Size)

Group Allocation

Select the option that describes the constraints on *N1* or *N2* or both.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. Since you are solving for both sample sizes at once, no additional sample size parameters need to be entered.

- **Enter N2, solve for N1**

  Select this option when you wish to fix *N2* at some value (or values), and then solve only for *N1*. Please note that for some values of *N2*, there may not be a value of *N1* that is large enough to obtain the desired power.

- **Enter R = N2/N1, solve for N1 and N2**

  For this choice, you set a value for the ratio of *N2* to *N1*, and then PASS determines the needed *N1* and *N2*, with this ratio, to obtain the desired power. An equivalent representation of the ratio, *R*, is

  $$N2 = R * N1.$$

- **Enter percentage in Group 1, solve for N1 and N2**

  For this choice, you set a value for the percentage of the total sample size that is in Group 1, and then PASS determines the needed *N1* and *N2* with this percentage to obtain the desired power.

N2 (Sample Size, Group 2)

This option is displayed if Group Allocation = "Enter N2, solve for N1"

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter R = N2/N1, solve for N1 and N2."

R is the ratio of *N2* to *N1*. That is,

$$R = N2 / N1.$$

Use this value to fix the ratio of *N2* to *N1* while solving for *N1* and *N2*. Only sample size combinations with this ratio are considered.

N2 is related to *N1* by the formula:

$$N2 = [R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1 (e.g., *N1* = 10 and *N2* = 20, or *N1* = 50 and *N2* = 100).

R must be greater than 0. If *R* < 1, then *N2* will be less than *N1*; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter percentage in Group 1, solve for N1 and N2."

Use this value to fix the percentage of the total sample size allocated to Group 1 while solving for *N1* and *N2*. Only sample size combinations with this Group 1 percentage are considered. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single or a series of values.

Sample Size (When Not Solving for Sample Size)

Group Allocation

Select the option that describes how individuals in the study will be allocated to Group 1 and to Group 2.

The options are

- **Equal (N1 = N2)**
  This selection is used when you wish to have equal sample sizes in each group. A single per group sample size will be entered.
- **Enter N1 and N2 individually**
  This choice permits you to enter different values for *N1* and *N2*.
- **Enter N1 and R, where N2 = R * N1**
  Choose this option to specify a value (or values) for *N1*, and obtain *N2* as a ratio (multiple) of *N1*.
- **Enter total sample size and percentage in Group 1**
  Choose this option to specify a value (or values) for the total sample size (*N*), obtain *N1* as a percentage of *N*, and then *N2* as *N* - *N1*.

Sample Size Per Group

This option is displayed only if Group Allocation = "Equal (N1 = N2)."

The Sample Size Per Group is the number of items or individuals sampled from each of the Group 1 and Group 2 populations. Since the sample sizes are the same in each group, this value is the value for *N1*, and also the value for *N2*.

The Sample Size Per Group must be ≥ 2. You can enter a single value or a series of values.

N1 (Sample Size, Group 1)

*This option is displayed if Group Allocation = "Enter N1 and N2 individually" or "Enter N1 and R, where N2 = R * N1."*

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed only if Group Allocation = "Enter N1 and N2 individually."

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

*This option is displayed only if Group Allocation = "Enter N1 and R, where N2 = R * N1."*

R is the ratio of *N2* to *N1*. That is,

$$R = N2/N1$$

Use this value to obtain *N2* as a multiple (or proportion) of *N1*.

N2 is calculated from *N1* using the formula:

$$N2=[R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1.

R must be greater than 0. If *R* < 1, then *N2* will be less than N1; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single value or a series of values.

Total Sample Size (N)

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This is the total sample size, or the sum of the two group sample sizes. This value, along with the percentage of the total sample size in Group 1, implicitly defines *N1* and *N2*.

The total sample size must be greater than one, but practically, must be greater than 3, since each group sample size needs to be at least 2.

You can enter a single value or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This value fixes the percentage of the total sample size allocated to Group 1. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single value or a series of values.

M (Measurements Per Subject)

Enter one or more values for M: the number of repeated measurements made on each subject.

You can enter a single value such as 2, a series of values such as 2 3 4, or 2 to 8 by 1.

The range is $M \geq 2$.

Effect Size

R0 (H0 Variance Ratio)

Enter one or more values for the ratio of the two within-subject variances assumed by the null hypothesis, H0. The sample variance ratio is compared to this value when conducting the test.

The usual equality test assumes this value is one. This procedure allows you to enter values other than one.

The range of possible values is $R0 > 0$. $R0 \neq R1$.

R1 (Actual Variance Ratio)

Enter one or more values for within-subject variance ratio assumed by the alternative hypothesis. This is the value of $\sigma^2 w_1 / \sigma^2 w_2$ at which the power is calculated.

The range of possible values is $R0 > 0$. $R0 \neq R1$.

Example 1 – Finding Sample Size

A company has developed a generic drug for treating rheumatism and wants to compare it to the standard drug in terms of the within-subject variability. A parallel-group design with replicates will be used to test the inequality using a two-sided test.

Company researchers set the variance ratio under the null hypothesis to 0.75, the significance level to 0.05, the power to 0.90, M to 2 or 3, and the actual variance ratio values between 0.5 and 1.2. They want to investigate the range of required sample size values assuming that the two group sample sizes are equal.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Non-Unity Null Tests for the Ratio of Within-Subject Variances in a Parallel Design** procedure window. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**Two-Sided (H1: $\sigma^2 w_1$ / $\sigma^2 w_2$ ≠ R0)**
Power	**0.9**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
M (Measurements Per Subject)	**2 3**
R0 (H0 Variance Ratio)	**0.75**
R1 (Actual Variance Ratio)	**0.5 0.6 0.9 1 1.1 1.2**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

H0: σ²[W1]/σ²[W2] = R0 vs. H1: σ²[W1]/σ²[W2] ≠ R0

Actual Power	Group 1 Sample Size N1	Group 2 Sample Size N2	N	Measures Per Subject M	H0 Variance Ratio R0	Actual Variance Ratio R1	Alpha
0.9004	257	257	514	2	0.750	0.500	0.050
0.9015	129	129	258	3	0.750	0.500	0.050
0.9003	846	846	1692	2	0.750	0.600	0.050
0.9003	423	423	846	3	0.750	0.600	0.050
0.9001	1266	1266	2532	2	0.750	0.900	0.050
0.9001	633	633	1266	3	0.750	0.900	0.050
0.9001	509	509	1018	2	0.750	1.000	0.050
0.9006	255	255	510	3	0.750	1.000	0.050
0.9005	288	288	576	2	0.750	1.100	0.050
0.9005	144	144	288	3	0.750	1.100	0.050
0.9011	192	192	384	2	0.750	1.200	0.050
0.9011	96	96	192	3	0.750	1.200	0.050

References

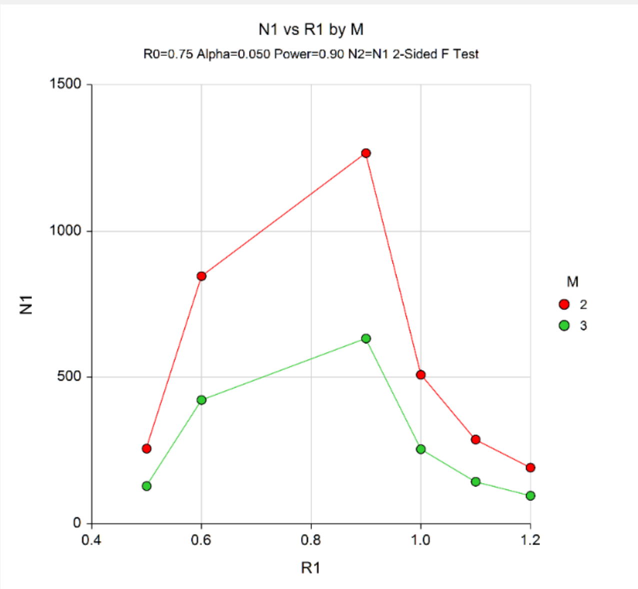
Chow, S.C., Shao, J., Wang, H., and Lokhnygina, Y. 2018. Sample Size Calculations in Clinical Research, Third Edition. Taylor & Francis/CRC. Boca Raton, Florida.
Chow, S.C. and Liu, J.P. 2014. Design and Analysis of Clinical Trials, Third Edition. John Wiley & Sons. Hoboken, New Jersey.

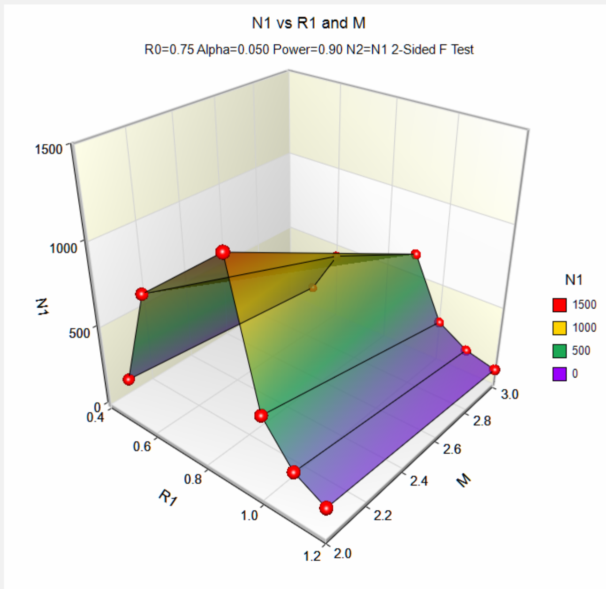
Report Definitions
Actual Power is the actual power achieved. Because N1 and N2 are discrete, this value is usually slightly larger than the target power.
N1 is the number of subjects from group 1. Each subject is measured M times.
N2 is the number of subjects from group 2. Each subject is measured M times.
N is the total number of subjects. N = N1 + N2.
M is the number of times each subject is measured.
R0 is the within-subject variance ratio used to define the null hypothesis, H0.
R1 is the value of the within-subject variance ratio at which the power is calculated.
Alpha is the probability of rejecting a true null hypothesis, H0.

Summary Statements
A study is being conducted to compare the within-subject variance of the treatment group to a reference group using a two-sided hypothesis test. Group sample sizes of 257 and 257 achieve 90% power to reject the null hypothesis at a significance level of 0.050. The variance ratio assumed by the null hypothesis is 0.750. The variance ratio at which the power is calculated is 0.500. Each subject is measured 2 times.

This report gives the sample sizes for the indicated scenarios.

Plot Section





These plots show the relationship between sample size, R1, and M.

Example 2 – Validation using Chow et al. (2018)

The following example is shown in Chow *et al.* (2018) page 195.

Find the sample size when R0 is 1.21, the significance level to 0.05, M is 3, the power is 0.8, and R1 is 0.44444444. They obtained N1 = N2 = 13. Their example is for a non-inferiority test.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Non-Unity Null Tests for the Ratio of Within-Subject Variances in a Parallel Design** procedure window. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**One-Sided (H1: $\sigma^2 w_1/\sigma^2 w_2$ < R0)**
Power	**0.80**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
M (Measurements Per Subject)	**3**
R0 (H0 Variance Ratio)	**1.21**
R1 (Actual Variance Ratio)	**0.44444444**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

H0: σ²[W1]/σ²[W2] ≤ R0 vs. H1: σ²[W1]/σ²[W2] > R0

Actual Power	Group 1 Sample Size N1	Group 2 Sample Size N2	N	Measures Per Subject M	H0 Variance Ratio R0	Actual Variance Ratio R1	Alpha
0.8072	13	13	26	3	1.210	0.444	0.050

The sample sizes match Chow et al. (2018).