

Chapter 414

One-Sample Z-Tests

Introduction

The one-sample z -test is used to test whether the mean of a population is greater than, less than, or not equal to a specific value. Because the standard normal distribution is used to calculate critical values for the test, this test is often called the one-sample z -test. The z -test assumes that the population standard deviation is known.

Other PASS Procedures for Testing One Mean or Median

Procedures in **PASS** are primarily built upon the testing methods, test statistic, and test assumptions that will be used when the analysis of the data is performed. You should check to identify that the test procedure described below in the Test Procedure section matches your intended procedure. If your assumptions or testing method are different, you may wish to use one of the other one-sample procedures available in **PASS**—the One-Sample T-Tests and the nonparametric Wilcoxon Signed-Rank Test procedures. The methods, statistics, and assumptions for those procedures are described in the associated chapters.

If you wish to show that the mean of a population is larger (or smaller) than a reference value by a specified amount, you should use one of the clinical superiority procedures for comparing means. Non-inferiority, equivalence, and confidence interval procedures are also available.

The Statistical Hypotheses

In the usual z -test setting, the null (H_0) and alternative (H_1) hypotheses for two-sided tests are defined as

$$H_0: \mu = \mu_0 \quad \text{versus} \quad H_1: \mu \neq \mu_0$$

Rejecting H_0 implies that the mean is not equal to the value μ_0 . The hypotheses for one-sided upper-tail tests are

$$H_0: \mu \leq \mu_0 \quad \text{versus} \quad H_1: \mu > \mu_0$$

Rejecting H_0 implies that the mean is larger than the value μ_0 . This test is called an *upper-tail test* because H_0 is rejected in samples in which the sample mean is larger than μ_0 .

The *lower-tail test* is

$$H_0: \mu \geq \mu_0 \quad \text{versus} \quad H_1: \mu < \mu_0$$

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It will be convenient to adopt the following specialized notation for the discussion of these tests.

<u>Parameter</u>	<u>PASS Input/Output</u>	<u>Interpretation</u>
μ	μ	<i>Population mean.</i> If the data are paired differences, this is the mean of those differences. This parameter will be estimated by the study.
μ_1	$\mu 1$	<i>Actual population mean at which power is calculated.</i> This is the assumed population mean used in all calculations.
μ_0	$\mu 0$	<i>Reference value.</i> Usually, this is the mean of a reference population. If the data are paired differences, this is the hypothesized value of the mean difference.
δ	δ	<i>Population difference.</i> This is the value of $\mu - \mu_0$, the difference between the population mean and the reference value. This parameter will be estimated by the study.
δ_1	$\delta 1$	<i>Actual difference at which power is calculated.</i> This is the value of $\mu_1 - \mu_0$, the assumed difference between the mean and the reference value for power calculations.

Assumptions for One-Sample Tests

This section describes the assumptions that are made when you use one of the one-sample tests. The key assumption relates to normality or non-normality of the data. One of the reasons for the popularity of the t -test is its robustness in the face of assumption violation. However, if an assumption is not met even approximately, the significance levels and the power of the t -test are invalidated. Unfortunately, in practice it often happens that several assumptions are not met. This makes matters even worse! Hence, take the steps to check the assumptions before you make important decisions based on these tests.

One-Sample Z-Test Assumptions

The assumptions of the one-sample z -test are:

1. The data are continuous (not discrete).
2. The data follow the normal probability distribution.
3. The sample is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.
4. The population standard deviation is known.

One-Sample T-Test Assumptions

The assumptions of the one-sample t -test are:

1. The data are continuous (not discrete).
2. The data follow the normal probability distribution.
3. The sample is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.

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Paired T-Test Assumptions

The assumptions of the paired t -test are:

1. The data are continuous (not discrete).
2. The data, i.e., the differences for the matched-pairs, follow a normal probability distribution.
3. The sample of pairs is a simple random sample from its population. Each individual in the population has an equal probability of being selected in the sample.

Wilcoxon Signed-Rank Test Assumptions

The assumptions of the Wilcoxon signed-rank test are as follows (note that the difference is between a data value and the hypothesized median or between the two data values of a pair):

1. The differences are continuous (not discrete).
2. The distribution of each difference is symmetric.
3. The differences are mutually independent.
4. The differences all have the same median.
5. The measurement scale is at least interval.

Limitations

There are few limitations when using these tests. Sample sizes may range from a few to several hundred. If your data are discrete with at least five unique values, you can often ignore the continuous variable assumption. Perhaps the greatest restriction is that your data come from a random sample of the population. If you do not have a random sample, your significance levels will probably be incorrect.

One-Sample Z-Test Statistic

The one-sample z -test assumes that the data are a simple random sample from a population of normally-distributed values that all have the same mean and variance (known). This assumption implies that the data are continuous and their distribution is symmetric. The calculation of the z -test proceeds as follows

$$z = \frac{\bar{X} - A}{\sigma/\sqrt{n}}$$

where

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$$

and A is the value of the mean hypothesized by the null hypothesis that incorporates both μ_0 and M_{NI} .

The significance of the test statistic is determined by computing the p-value. If this p-value is less than a specified level (usually 0.05), the hypothesis is rejected. Otherwise, no conclusion can be reached.

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Power Calculation for the One-Sample Z-Test

When the standard deviation is known, the power is calculated as follows for a directional alternative (one-tailed test) in which $\mu_1 > \mu_0$.

1. Find z_α such that $1 - \Phi(z_\alpha) = \alpha$, where $\Phi(x)$ is the area to the left of x under the standardized normal curve.
2. Calculate: $X_1 = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$.
3. Calculate: $z_1 = \frac{X_1 - \mu_1}{\frac{\sigma}{\sqrt{n}}}$.
4. Power = $1 - \Phi(z_1)$.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be calculated from the values of the other parameters. Under most conditions, you would select either *Power* or *Sample Size*.

Select *Sample Size* when you want to determine the sample size needed to achieve a given power and alpha error level.

Select *Power* when you want to calculate the power of an experiment that has already been run.

Test

Alternative Hypothesis

Specify the alternative hypothesis of the test. Since the null hypothesis is the opposite of the alternative, specifying the alternative is all that is needed. Usually, the two-tailed (\neq) option is selected.

The options containing only $<$ or $>$ are one-tailed tests. When you choose one of these options, you must be sure that the input parameters match this selection.

Possible selections are:

- **Two-Sided (H1: $\mu \neq \mu_0$)**

This is the most common selection. It yields the *two-tailed z-test*. Use this option when you are testing whether the means are different but you do not want to specify beforehand which mean is larger. Many scientific journals require two-tailed tests.

- **One-Sided (H1: $\mu < \mu_0$)**

This option yields a *one-tailed z-test*. Use it when you are only interested in the case in which μ is less than μ_0 .

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- **One-Sided (H1: $\mu > \mu_0$)**

This option yields a *one-tailed z-test*. Use it when you are only interested in the case in which μ is greater than μ_0 .

Population Size

This is the number of subjects in the population. Usually, you assume that samples are drawn from a very large (infinite) population. Occasionally, however, situations arise in which the population of interest is of limited size. In these cases, appropriate adjustments must be made.

When a finite population size is specified, the standard deviation is reduced according to the formula:

$$\sigma_1^2 = \left(1 - \frac{n}{N}\right) \sigma^2$$

where n is the sample size, N is the population size, σ is the original standard deviation, and σ_1 is the new standard deviation.

The quantity n/N is often called the sampling fraction. The quantity $\left(1 - \frac{n}{N}\right)$ is called the *finite population correction factor*.

Power and Alpha

Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus Beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected.

Values must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01*.

Sample Size

N (Sample Size)

This option specifies one or more values of the sample size, the number of individuals in the study. This value must be an integer greater than one. Note that you may enter a list of values using the syntax *50,100,150,200,250* or *50 to 250 by 50*.

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Effect Size – Means

μ_0 (Null or Baseline Mean)

Enter a value for the population mean under the null hypothesis. This is the reference or baseline mean.

Only the difference between μ_1 and μ_0 is used in the calculations.

μ_1 (Actual Mean)

Enter a value (or range of values) for the actual population mean at which power and sample size are calculated.

Only the difference between μ_1 and μ_0 is used in the calculations.

Effect Size – Standard Deviation

σ (Standard Deviation)

This option specifies one or more values of the known standard deviation. This must be a positive value.

PASS includes a special module for estimating the standard deviation. This module may be loaded by pressing the *SD* button. Refer to the Standard Deviation Estimator chapter for further details.

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Example 1 – Power after a Study

This example will cover the situation in which you are calculating the power of a z -test on data that have already been collected and analyzed. For example, you might be playing the role of a reviewer, looking at the power of a z -test from a study you are reviewing. In this case, you would not vary the means or sample size since they are given by the experiment. Instead, you investigate the power of the significance tests. You might look at the impact of different alpha values on the power.

Suppose an experiment involving 100 individuals yields the following summary statistics:

Hypothesized mean (μ_0)	100.0
Sample mean (μ_1)	110.0
Sample size	100

Given the above data, analyze the power of a z -test which tests the hypothesis that the population mean is 100 versus the alternative hypothesis that the population mean is 110. Consider the power at significance levels 0.01, 0.05, 0.10 and sample sizes 20 to 120 by 20. The standard deviation is known to be 40.

Note that we have set μ_1 equal to the sample mean. In this case, we are studying the power of the z -test for a mean difference the size of that found in the experimental data.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **One-Sample Z-Tests** procedure window by expanding **Means**, then **One Mean**, then clicking on **Z-Test (Inequality)**, and then clicking on **One-Sample Z-Tests**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Power
Alternative Hypothesis	Two-Sided (H1: $\mu \neq \mu_0$)
Population Size	Infinite
Alpha	0.01 0.05 0.10
N (Sample Size)	20 to 120 by 20
μ_0 (Null or Baseline Mean)	100
μ_1 (Actual mean)	110
σ (Standard Deviation)	40

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Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Hypotheses: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$

Power	N	μ_0	μ_1	Diff $\mu_1 - \mu_0$	σ	Effect Size	Alpha	Beta
0.07256	20	100.0	110.0	10.0	40.0	0.250	0.010	0.92744
0.15996	40	100.0	110.0	10.0	40.0	0.250	0.010	0.84004
0.26130	60	100.0	110.0	10.0	40.0	0.250	0.010	0.73870
0.36702	80	100.0	110.0	10.0	40.0	0.250	0.010	0.63298
0.46978	100	100.0	110.0	10.0	40.0	0.250	0.010	0.53022
0.56466	120	100.0	110.0	10.0	40.0	0.250	0.010	0.43534

(report continues)

Report Definitions

Power is the probability of rejecting a false null hypothesis. It should be close to one.
 N is the size of the sample drawn from the population. To conserve resources, it should be small.
 μ_0 is the value of the population mean under the null hypothesis.
 μ_1 is the actual value of the population mean at which power and sample size are calculated.
 $\mu_1 - \mu_0$ is the difference between the actual and null means.
 σ is the standard deviation of the population. It measures the variability in the population.
 Effect Size = $|\mu_1 - \mu_0|/\sigma$ is the relative magnitude of the effect.
 Alpha is the probability of rejecting a true null hypothesis. It should be small.
 Beta is the probability of accepting a false null hypothesis. It should be small.

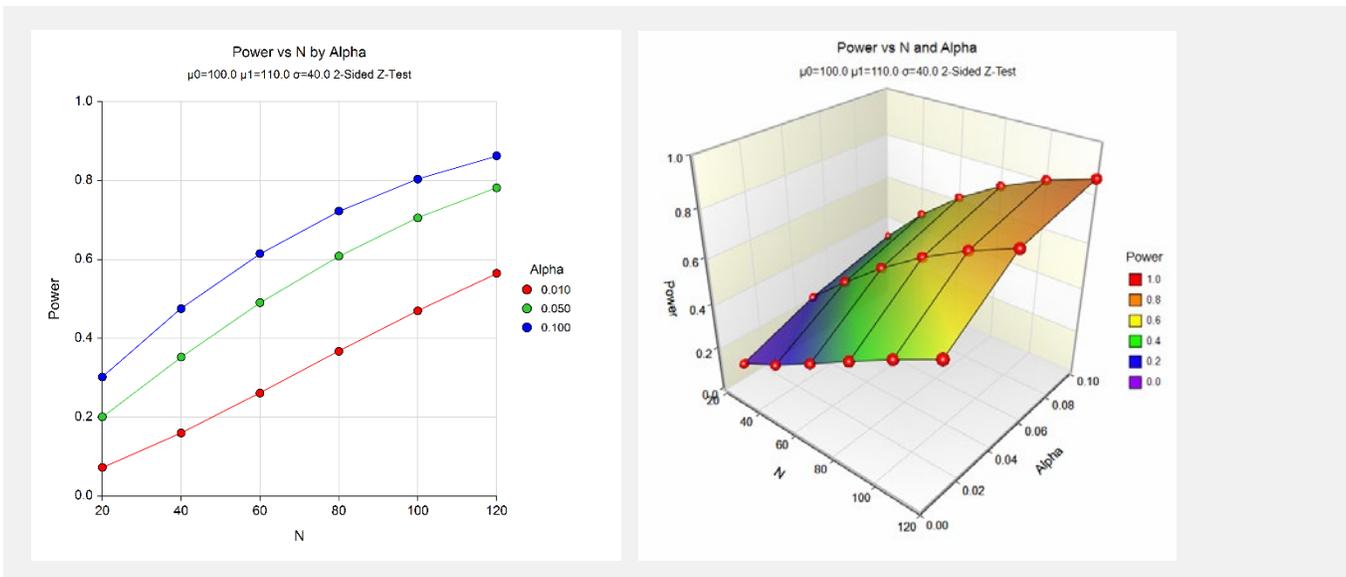
Summary Statements

A sample size of 20 achieves 7% power to detect a difference of 10.0 between the actual mean of 110.0 and the null hypothesized mean of 100.0 with a known standard deviation of 40.0 and with a significance level (alpha) of 0.010 using a two-sided one-sample z-test.

This report shows the values of each of the parameters, one scenario per row. The values of power and beta were calculated from the other parameters.

The definitions of each column are given in the Report Definitions section.

Plots Section



These plots show the relationship between sample size and power for various values of alpha.

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Example 2 – Finding the Sample Size

This example will consider the situation in which you are planning a study that will use the one-sample z -test and want to determine an appropriate sample size. This example is more subjective than the first because you now have to obtain estimates of all the parameters. In the first example, these estimates were provided by the data.

In studying deaths from SIDS (Sudden Infant Death Syndrome), one hypothesis put forward is that infants dying of SIDS weigh less than normal at birth. Suppose the average birth weight of infants is 3300 grams with a known standard deviation of 663 grams. Use an alpha of 0.05 and power of both 0.80 and 0.90. How large a sample of SIDS infants will be needed to detect a drop in average weight of 25%? Of 10%? Of 5%? Note that applying these percentages to the average weight of 3300 yields 2475, 2970, and 3135.

Although a one-sided hypothesis is being considered, sample size estimates will assume a two-sided alternative to keep the research design in line with other studies.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **One-Sample Z-Tests** procedure window by expanding **Means**, then **One Mean**, then clicking on **Z-Test (Inequality)**, and then clicking on **One-Sample Z-Tests**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Sample Size
Alternative Hypothesis	Two-Sided (H1: $\mu \neq \mu_0$)
Population Size	Infinite
Power	0.80 0.90
Alpha	0.05
μ_0 (Null or Baseline Mean)	3300
μ_1 (Actual mean)	2475 2970 3135
σ (Standard Deviation)	663

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results								
Hypotheses: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$								
Power	N	μ_0	μ_1	Diff $\mu_1 - \mu_0$	σ	Effect Size	Alpha	Beta
0.86171	6	3300.0	2475.0	-825.0	663.0	1.244	0.050	0.13829
0.90861	7	3300.0	2475.0	-825.0	663.0	1.244	0.050	0.09139
0.80391	32	3300.0	2970.0	-330.0	663.0	0.498	0.050	0.19609
0.90387	43	3300.0	2970.0	-330.0	663.0	0.498	0.050	0.09613
0.80085	127	3300.0	3135.0	-165.0	663.0	0.249	0.050	0.19915
0.90058	170	3300.0	3135.0	-165.0	663.0	0.249	0.050	0.09942

This report shows the values of each of the parameters, one scenario per row. Since there were three values of μ_1 and two values of power, there are a total of six rows in the report.

We were solving for the sample size, N . Notice that the increase in sample size seems to be most directly related to the difference between the two means. The difference in beta values does not seem to be as influential, especially at the smaller sample sizes.

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Note that even though we set the power values at 0.8 and 0.9, these are not the power values that were achieved. This happens because N can only take on integer values. The program selects the first value of N that gives at least the values of alpha and power that were desired.

Example 3 – Finding the Minimum Detectable Difference

This example will consider the situation in which you want to determine how small of a difference between the two means can be detected by the z -test with specified values of the other parameters.

Continuing with the previous example, suppose about 50 SIDS deaths occur in a particular area per year. Using 50 as the sample size, 0.05 as alpha, and 0.80 as power, how large of a difference between the means is detectable?

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **One-Sample Z-Tests** procedure window by expanding **Means**, then **One Mean**, then clicking on **Z-Test (Inequality)**, and then clicking on **One-Sample Z-Tests**. You may then make the appropriate entries as listed below, or open **Example 3** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	μ_1 (Search < μ_0)
Alternative Hypothesis	Two-Sided ($H_1: \mu \neq \mu_0$)
Population Size	Infinite
Power	0.80
Alpha	0.05
N (Sample Size)	50
μ_0 (Null or Baseline Mean)	3300
σ (Standard Deviation)	663

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results									
Hypotheses: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$									
Power	N	μ_0	μ_1	Diff $\mu_1 - \mu_0$	σ	Effect Size	Alpha	Beta	
0.80000	50	3300.0	3037.3	-262.7	663.0	0.396	0.050	0.20000	

With a sample of 50, a difference of $3037.3 - 3300 = -262.7$ would be detectable.

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Example 4 – Validation using Chow, Shao, Wang, and Lokhnygina (2018)

Chow, Shao, Wang, and Lokhnygina (2018) presents an example on pages 45 and 46 of a two-sided one-sample z-test sample size calculation in which $\mu_0 = 1.5$, $\mu_1 = 2.0$, $\sigma = 1.0$, $\alpha = 0.05$, and $\text{power} = 0.80$. They obtain a sample size of 32.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **One-Sample Z-Tests** procedure window by expanding **Means**, then **One Mean**, then clicking on **Z-Test (Inequality)**, and then clicking on **One-Sample Z-Tests**. You may then make the appropriate entries as listed below, or open **Example 4** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Sample Size
Alternative Hypothesis	Two-Sided (H1: $\mu \neq \mu_0$)
Population Size	Infinite
Power	0.80
Alpha	0.05
μ_0 (Null or Baseline Mean)	1.5
μ_1 (Actual Mean)	2
σ (Standard Deviation)	1

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results								
Hypotheses: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$								
Power	N	μ_0	μ_1	Diff $\mu_1 - \mu_0$	σ	Effect Size	Alpha	Beta
0.80743	32	1.5	2.0	0.5	1.0	0.500	0.050	0.19257

The sample size of 32 matches Chow, Shao, Wang, and Lokhnygina (2018) exactly.

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Example 5 – Validation using Machin (1997)

Machin, Campbell, Fayers, and Pinol (1997) page 37 presents an example of a one-sample t -test in which $\mu_0 = 0.0$, $\mu_1 = 0.2$, $\sigma = 1.0$, $\alpha = 0.05$, and $\beta = 0.20$. They obtain a sample size of 199. The z -test should give a similar but slightly lower result because the normal distribution approximates the t distribution very well at this sample size.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **One-Sample Z-Tests** procedure window by expanding **Means**, then **One Mean**, then clicking on **Z-Test (Inequality)**, and then clicking on **One-Sample Z-Tests**. You may then make the appropriate entries as listed below, or open **Example 5** by going to the **File** menu and choosing **Open Example Template**.

<u>Option</u>	<u>Value</u>
Design Tab	
Solve For	Sample Size
Alternative Hypothesis	Two-Sided (H1: $\mu \neq \mu_0$)
Population Size	Infinite
Power	0.80
Alpha	0.05
μ_0 (Null or Baseline Mean)	0
μ_1 (Actual mean)	0.2
σ (Standard Deviation)	1

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results								
Hypotheses: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$								
Power	N	μ_0	μ_1	Diff $\mu_1 - \mu_0$	σ	Effect Size	Alpha	Beta
0.80155	197	0.0	0.2	0.2	1.0	0.200	0.050	0.19845

The sample size of 197 is very close to and just less than Machin's result for the t -test.