Chapter 655

Tests for Two Variances

Introduction

Occasionally, researchers are interested in comparing the variances (or standard deviations) of two groups rather than their means. This module calculates the sample sizes and performs power analyses for hypothesis tests concerning two variances.

Technical Details

Assuming that variables *X1* and *X2* are normally distributed variances σ_1^2 and σ_2^2 (the means are ignored), the distribution of the ratio of the sample variances follows the *F* distribution. That is,

$$F = \frac{s_1^2}{s_2^2}$$

is distributed as an *F* random variable with $N_1 - 1$ and $N_2 - 1$ degrees of freedom. The sample statistic, s_j^2, is calculated as follows

$$s_j^2 = \frac{\sum_{i=1}^{N} \left(X_{ji} - \overline{X}_j \right)^2}{N_j - 1}.$$

The power or sample size of a hypothesis test about the variance can be calculated using the appropriate one of the following three formulas:

Case 1: $H_0: \sigma_1^2 = \sigma_2^2$ versus $H_a: \sigma_1^2 \neq \sigma_2^2$

$$\beta = P\left(\frac{\sigma_2^2}{\sigma_1^2} F_{\alpha/2, N_1-1, N_2-1} < F < \frac{\sigma_2^2}{\sigma_1^2} F_{1-\alpha/2, N_1-1, N_2-1} \right)$$

Case 2: $H_0: \sigma_1^2 \leq \sigma_2^2$ versus $H_a: \sigma_1^2 > \sigma_2^2$

$$\beta = P\left(F < \frac{\sigma_2^2}{\sigma_1^2} F_{1-\alpha, N_1-1, N_2-1} \right)$$

Case 3: $H_0: \sigma_1^2 \geq \sigma_2^2$ versus $H_a: \sigma_1^2 < \sigma_2^2$

$$\beta = 1 - P\left(F < \frac{\sigma_2^2}{\sigma_1^2} F_{\alpha, N_1-1, N_2-1}\right)$$

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters.

Test

Alternative Hypothesis

This option specifies the alternative hypothesis. This implicitly specifies the direction of the hypothesis test. The null hypothesis is always $H_0: V_1 = V_2$.

Note that the alternative hypothesis enters into power calculations by specifying the rejection region of the hypothesis test. Its accuracy is critical.

Possible selections are:

- **Ha: V1 ≠ V2**

  This selection yields a *two-tailed* test. Use this option when you are testing whether the variances are different but you do not want to specify beforehand which variance is larger.

- **Ha: V1 > V2**

  The options yields a *one-tailed* test. Use it when you are only interested in the case in which *V2* is less than *V1*.

- **Ha: V1 < V2**

  This option yields a *one-tailed* test. Use it when you are only interested in the case in which *V2* is greater than *V1*.

Power and Alpha

Power

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis, and is equal to one minus Beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as *0.8 to 0.95 by 0.05* may be entered.

Alpha

This option specifies one or more values for the probability of a type-I error. A type-I error occurs when a true null hypothesis is rejected.

Values must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as *0.01 0.05 0.10* or *0.01 to 0.10 by 0.01*.

Sample Size (When Solving for Sample Size)

Group Allocation

Select the option that describes the constraints on *N1* or *N2* or both.

The options are

- **Equal (N1 = N2)**

  This selection is used when you wish to have equal sample sizes in each group. Since you are solving for both sample sizes at once, no additional sample size parameters need to be entered.

- **Enter N1, solve for N2**

  Select this option when you wish to fix *N1* at some value (or values), and then solve only for *N2*. Please note that for some values of *N1*, there may not be a value of *N2* that is large enough to obtain the desired power.

- **Enter N2, solve for N1**

  Select this option when you wish to fix *N2* at some value (or values), and then solve only for *N1*. Please note that for some values of *N2*, there may not be a value of *N1* that is large enough to obtain the desired power.

- **Enter R = N2/N1, solve for N1 and N2**

  For this choice, you set a value for the ratio of *N2* to *N1*, and then PASS determines the needed *N1* and *N2*, with this ratio, to obtain the desired power. An equivalent representation of the ratio, *R*, is

  $$N2 = R * N1.$$

- **Enter percentage in Group 1, solve for N1 and N2**

  For this choice, you set a value for the percentage of the total sample size that is in Group 1, and then PASS determines the needed *N1* and *N2* with this percentage to obtain the desired power.

N1 (Sample Size, Group 1)

This option is displayed if Group Allocation = "Enter N1, solve for N2"

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed if Group Allocation = "Enter N2, solve for N1"

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

This option is displayed only if Group Allocation = "Enter R = N2/N1, solve for N1 and N2."

R is the ratio of *N2* to *N1*. That is,

$$R = N2 / N1.$$

Use this value to fix the ratio of *N2* to *N1* while solving for *N1* and *N2*. Only sample size combinations with this ratio are considered.

N2 is related to *N1* by the formula:

$$N2 = [R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting *R* = 2.0 results in a Group 2 sample size that is double the sample size in Group 1 (e.g., *N1* = 10 and *N2* = 20, or *N1* = 50 and *N2* = 100).

R must be greater than 0. If *R* < 1, then *N2* will be less than *N1*; if *R* > 1, then *N2* will be greater than *N1*. You can enter a single or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter percentage in Group 1, solve for N1 and N2."

Use this value to fix the percentage of the total sample size allocated to Group 1 while solving for *N1* and *N2*. Only sample size combinations with this Group 1 percentage are considered. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single or a series of values.

Sample Size (When Not Solving for Sample Size)

Group Allocation

Select the option that describes how individuals in the study will be allocated to Group 1 and to Group 2.

The options are

- **Equal (N1 = N2)**
  This selection is used when you wish to have equal sample sizes in each group. A single per group sample size will be entered.
- **Enter N1 and N2 individually**
  This choice permits you to enter different values for *N1* and *N2*.
- **Enter N1 and R, where N2 = R * N1**
  Choose this option to specify a value (or values) for *N1*, and obtain *N2* as a ratio (multiple) of *N1*.
- **Enter total sample size and percentage in Group 1**
  Choose this option to specify a value (or values) for the total sample size (*N*), obtain *N1* as a percentage of *N*, and then *N2* as *N - N1*.

Sample Size Per Group

This option is displayed only if Group Allocation = "Equal (N1 = N2)."

The Sample Size Per Group is the number of items or individuals sampled from each of the Group 1 and Group 2 populations. Since the sample sizes are the same in each group, this value is the value for *N1*, and also the value for *N2*.

The Sample Size Per Group must be ≥ 2. You can enter a single value or a series of values.

N1 (Sample Size, Group 1)

*This option is displayed if Group Allocation = "Enter N1 and N2 individually" or "Enter N1 and R, where N2 = R * N1."*

N1 is the number of items or individuals sampled from the Group 1 population.

N1 must be ≥ 2. You can enter a single value or a series of values.

N2 (Sample Size, Group 2)

This option is displayed only if Group Allocation = "Enter N1 and N2 individually."

N2 is the number of items or individuals sampled from the Group 2 population.

N2 must be ≥ 2. You can enter a single value or a series of values.

R (Group Sample Size Ratio)

*This option is displayed only if Group Allocation = "Enter N1 and R, where N2 = R * N1."*

R is the ratio of *N2* to *N1*. That is,

$$R = N2/N1$$

Use this value to obtain *N2* as a multiple (or proportion) of *N1*.

N2 is calculated from *N1* using the formula:

$$N2=[R \times N1],$$

where the value *[Y]* is the next integer ≥ *Y*.

For example, setting R = 2.0 results in a Group 2 sample size that is double the sample size in Group 1.

R must be greater than 0. If R < 1, then *N2* will be less than N1; if R > 1, then *N2* will be greater than *N1*. You can enter a single value or a series of values.

Total Sample Size (N)

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This is the total sample size, or the sum of the two group sample sizes. This value, along with the percentage of the total sample size in Group 1, implicitly defines *N1* and *N2*.

The total sample size must be greater than one, but practically, must be greater than 3, since each group sample size needs to be at least 2.

You can enter a single value or a series of values.

Percent in Group 1

This option is displayed only if Group Allocation = "Enter total sample size and percentage in Group 1."

This value fixes the percentage of the total sample size allocated to Group 1. Small variations from the specified percentage may occur due to the discrete nature of sample sizes.

The Percent in Group 1 must be greater than 0 and less than 100. You can enter a single value or a series of values.

Effect Size

Scale

Specify whether *V1* and *V2* are variances or standard deviations.

V1 and V2

Enter one or more value(s) for the variances of the groups, σ_1^2 and σ_2^2. All entries must be greater than zero.

Note that since the ratio of these variances is all that is used in the power equations, you can specify the problem in terms of the variance ratio instead of the two variances. To do this, enter 1.0 for *V2* and enter the desired variance ratio in *V1*.

If Scale is *Standard Deviation* this value is the standard deviation rather than the variance.

Example 1 – Calculating the Power

A machine used to perform a particular analysis is to be replaced with a new type of machine if the new machine reduces the variance in the output by 50%. If the significance level is set to 0.05, calculate the power for sample sizes of 5, 10, 20, 35, 50, 90, 130, and 200.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for Two Variances** procedure window by expanding **Variances**, then clicking on **Two Variances**, and then clicking on **Tests for Two Variances**. You may then make the appropriate entries as listed below, or open **Example 1** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Power**
Alternative Hypothesis	**Ha: V1 > V2**
Alpha	**0.01 0.05**
Group Allocation	**Equal (N1 = N2)**
Sample Size Per Group	**5 10 20 35 50 90 130 200**
Scale	**Variance**
V1 (Variance of Group 1)	**1.0**
V2 (Variance of Group 2)	**0.5**

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results when H0: V1 = V2 versus Ha: V1 > V2

Power	N1	N2	N	V1	V2	Alpha
0.03438	5	5	10	1.0000	0.5000	0.010
0.07939	10	10	20	1.0000	0.5000	0.010
0.18712	20	20	40	1.0000	0.5000	0.010
0.36265	35	35	70	1.0000	0.5000	0.010
0.52621	50	50	100	1.0000	0.5000	0.010
0.82160	90	90	180	1.0000	0.5000	0.010
0.94426	130	130	260	1.0000	0.5000	0.010
0.99453	200	200	400	1.0000	0.5000	0.010
0.14344	5	5	10	1.0000	0.5000	0.050
0.25042	10	10	20	1.0000	0.5000	0.050
0.43104	20	20	40	1.0000	0.5000	0.050
0.63686	35	35	70	1.0000	0.5000	0.050
0.77651	50	50	100	1.0000	0.5000	0.050
0.94602	90	90	180	1.0000	0.5000	0.050
0.98850	130	130	260	1.0000	0.5000	0.050
0.99936	200	200	400	1.0000	0.5000	0.050

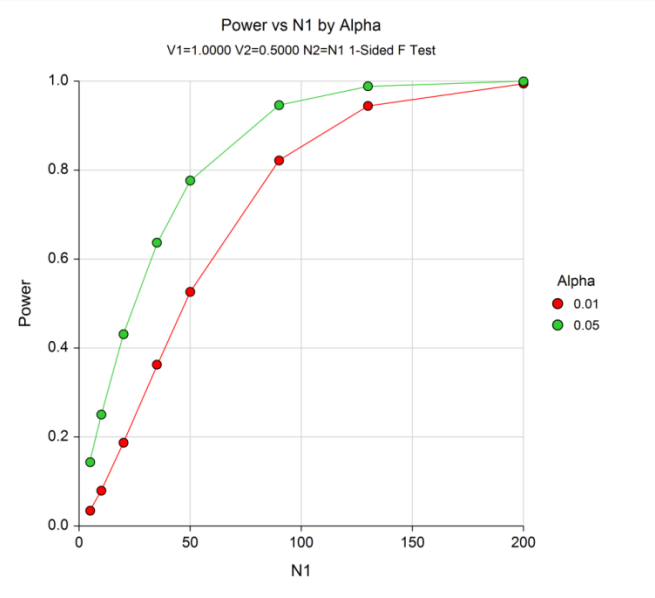
Tests for Two Variances

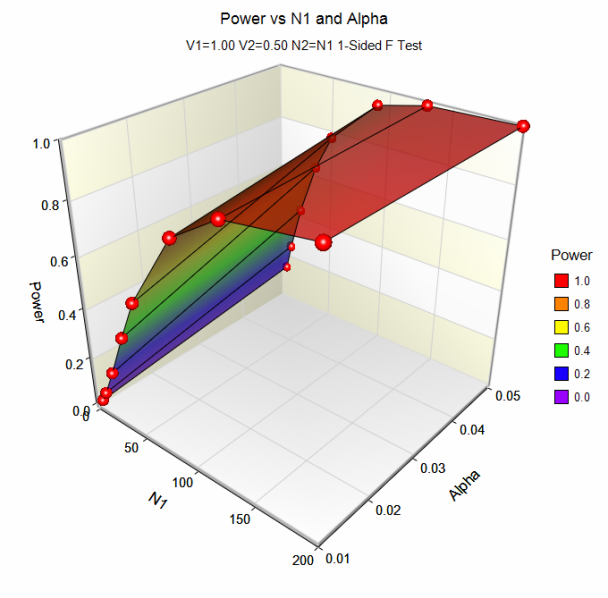
Report Definitions
Power is the probability of rejecting a false null hypothesis.
N1 and N2 are the number of items sampled from each population.
N is the total sample size, N1 + N2.
V1 is the value of the population variance of group 1.
V2 is the value of the population variance of group 2.
Alpha is the probability of rejecting a true null hypothesis.

Summary Statements
Group sample sizes of 5 and 5 achieve 3% power to detect a ratio of 2.0000 between the group one variance of 1.0000 and the group two variance of 0.5000 using a one-sided F test with a significance level (alpha) of 0.010000.

This report shows the calculated power for each scenario.

Plots Section





These plots show the power versus the sample size for the two significance levels. It is now easy to determine an appropriate sample size to meet both the alpha and beta objectives of the study.

Example 2 – Calculating Sample Size

Continuing with the previous example, the analyst wants to find the necessary sample sizes to achieve a power of 0.9 for two significance levels, 0.01 and 0.05, and for several variance ratio values of 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for Two Variances** procedure window by expanding **Variances**, then clicking on **Two Variances**, and then clicking on **Tests for Two Variances**. You may then make the appropriate entries as listed below, or open **Example 2** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**Ha: V1 > V2**
Power	**0.90**
Alpha	**0.01 0.05**
Group Allocation	**Equal (N1 = N2)**
Scale	**Variance**
V1 (Variance of Group 1)	**1.0**
V2 (Variance of Group 2)	**0.2 to 0.7 by 0.1**

Output

Click the Calculate button to perform the calculations and generate the following output.

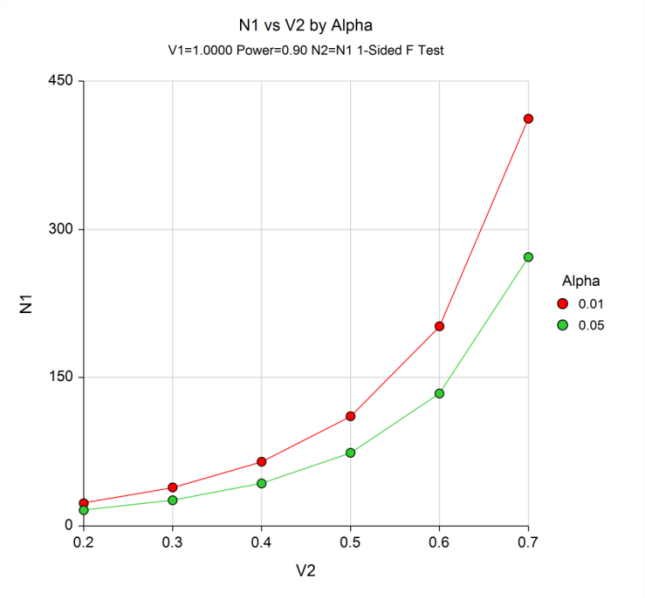
Numeric Results

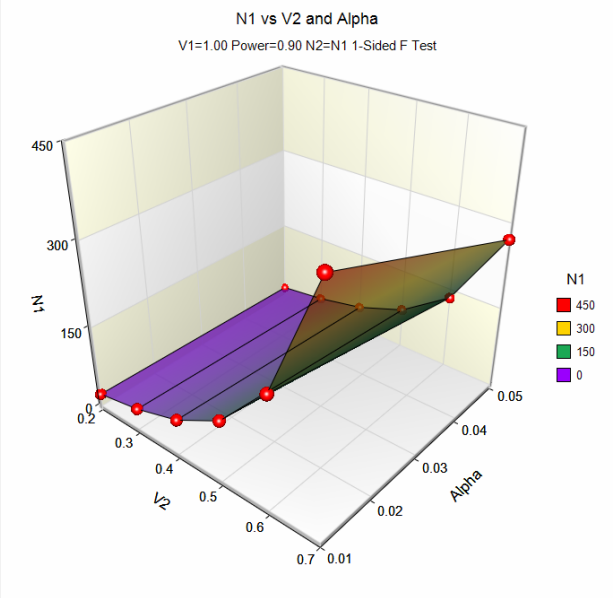
Numeric Results when H0: V1 = V2 versus Ha: V1 > V2

Target Power	Actual Power	N1	N2	N	V1	V2	Alpha
0.90	0.91111	23	23	46	1.0000	0.2000	0.010
0.90	0.91624	16	16	32	1.0000	0.2000	0.050
0.90	0.90777	39	39	78	1.0000	0.3000	0.010
0.90	0.90532	26	26	52	1.0000	0.3000	0.050
0.90	0.90408	65	65	130	1.0000	0.4000	0.010
0.90	0.90207	43	43	86	1.0000	0.4000	0.050
0.90	0.90132	111	111	222	1.0000	0.5000	0.010
0.90	0.90295	74	74	148	1.0000	0.5000	0.050
0.90	0.90045	202	202	404	1.0000	0.6000	0.010
0.90	0.90165	134	134	268	1.0000	0.6000	0.050
0.90	0.90042	412	412	824	1.0000	0.7000	0.010
0.90	0.90082	272	272	544	1.0000	0.7000	0.050

This report shows the necessary sample size for each scenario.

Plot Section





These plots show the necessary sample size for various values of *V2*. Note that as *V2* nears *V1*, the sample size is increased.

Example 3 – Validation using Davies (1971)

Davies (1971) page 41 presents an example with *V1* = 4, *V2* = 1, *Alpha* = 0.05, and *Power* = 0.99 in which the sample sizes, *N1* and *N2*, are calculated to be 36. We will run this example through **PASS**.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the **Tests for Two Variances** procedure window by expanding **Variances**, then clicking on **Two Variances**, and then clicking on **Tests for Two Variances**. You may then make the appropriate entries as listed below, or open **Example 3** by going to the **File** menu and choosing **Open Example Template**.

Option	Value
Design Tab	
Solve For	**Sample Size**
Alternative Hypothesis	**Ha: V1 > V2**
Power	**0.99**
Alpha	**0.05**
Group Allocation	**Equal (N1 = N2)**
Scale	**Variance**
V1 (Variance of Group 1)	**4**
V2 (Variance of Group 2)	**1**

Output

Click the Calculate button to perform the calculations and generate the following output.

Numeric Results

Numeric Results when H0: V1 = V2 versus Ha: V1 > V2

Target Power	Actual Power	N1	N2	N	V1	V2	Alpha
0.99	0.99142	36	36	72	4.0000	1.0000	0.050

PASS calculates N1 and N2 to be 36, which matches Davies' result.