Chapter 826

Tests of Mediation Effect in Poisson Regression

Introduction

This procedure computes power and sample size for a test of the mediation effect in a Poisson regression with a dependent (output) variable Y of counts and an independent (input) variable X. Interest focuses on the interrelationship between Y, X, and a third independent variable called the mediator M. The sample size calculations are based on the work of Vittinghoff, Sen, and McCulloch (2009). Note that their work has been extended in Vittinghoff and Neilands (2015). We are looking into adding those extensions in a later procedure.

Mediation Model

Vittinghoff, Sen, and McCulloch (2009) derived sample size formulas for testing the mediation effect based on testing the significance of $\beta_M$ in the Poisson regression model

$$\log \left( \frac{p}{1-p} \right) = \beta_0 + \beta_X X + \beta_M M.$$ 

They showed that testing $\beta_M = 0$ is equivalent to testing for a significant mediation effect. In addition to the notation above, they use $\rho_{XM}$ as the correlation between the independent variables X and M and $E(Y)$ to represent the marginal mean of Y.

Calculating the Power

Power calculations are based on the following assuming that if X is continuous, the variance of X is $\sigma_X^2$, or, if binary, the variance of X is $p_X(1-p_X)$ where $p_X = \text{Prob}(X = 1)$. Similar notation is used for the mediator variable M.

1. Determine the critical value $z_{1-\alpha}$ from the standard normal distribution where $\alpha$ is the probability of a type-I error.

2. a. If X is continuous and M continuous, use $z_\beta = \sqrt{N\sigma_M^2 \rho_{XM}^2 (1 - \rho_{XM}^2)} E(Y) - z_{1-\alpha}.$

   b. If X is binary and M continuous, use $z_\beta = \sqrt{N\beta_M^2 \rho_{XM}^2 (1 - \rho_{XM}^2)} E(Y) - z_{1-\alpha}.$

   c. If X is continuous and M binary, use $z_\beta = \sqrt{N\beta_M^2 (1 - \rho_{XM}^2)} E(Y) F1 - z_{1-\alpha}.$

   d. If X is binary and M binary, use $z_\beta = \sqrt{N\beta_M^2 (1 - \rho_{XM}^2)} E(Y) F2 - z_{1-\alpha}.$
where
\[ F_1 = \left[ GH \sigma^2_{X,M} \right] / \left[ (G + H)^2 \sigma^2_{X,M} + GH(\mu_1 - \mu_0)^2 \right] \]
\[ \mu_0 = -\rho_{XM} \sigma_X \sqrt{\frac{p_M}{1 - p_M}} \text{ when } M = 0 \]
\[ \mu_1 = \rho_{XM} \sigma_X \sqrt{\frac{1 - p_M}{p_M}} \text{ when } M = 1 \]
\[ \sigma^2_{X,M} = \sigma_X^2 (1 - \rho_{XM}^2) \]
\[ G = p_M \exp(\beta_X \mu_1 + \beta_M) \]
\[ H = (1 - p_M) \exp(\beta_X \mu_0) \]
\[ F_2 = \left[ BCD + BCE + BDE + CDE \right] / \left[ (B + C + D + E)(B + D)(C + E) \right] \]
\[ B = p_{00} \]
\[ C = p_{10} \exp(\beta_X) \]
\[ D = p_{01} \exp(\beta_M) \]
\[ E = p_{11} \exp(\beta_X + \beta_M) \]
\[ p_{11} = p_X p_M + \rho_{XM} \sqrt{p_X(1 - p_X)p_M(1 - p_M)} \text{ with } \rho_{XM} |0 < p_{11} < p_{X}^{PM}| \]
\[ p_{10} = p_X - p_{11} \]
\[ p_{01} = p_M - p_{11} \]
\[ p_{00} = 1 - p_{01} - p_{10} - p_{11} \]

3. Calculate: \( \text{Power} = \Phi(z_{\beta}) \).

Notes
1. Use \( \frac{\alpha}{2} \) instead of \( \alpha \) for two-sided test.

Procedure Options

This section describes the options that are specific to this procedure. These are located on the Design tab. For more information about the options of other tabs, go to the Procedure Window chapter.

Design Tab

The Design tab contains most of the parameters and options that you will be concerned with.

Solve For

Solve For

This option specifies the parameter to be solved for from the other parameters. Under most situations, you will select either Power or \( N(\text{Sample Size}) \).

Select Sample Size when you want to calculate the sample size needed to achieve a given power and alpha level.

Select Power when you want to calculate the power of an experiment.
Tests of Mediation Effect in Poisson Regression

**Test Direction**

**Alternative Hypothesis**

Specify whether the hypothesis test is one-sided or two-sided. When a two-sided test is selected, the value of alpha is automatically divided by two. A two-sided test requires alpha to be less than 0.50.

**Power and Alpha**

**Power**

This option specifies one or more values for power. Power is the probability of rejecting a false null hypothesis and is equal to one minus Beta. Beta is the probability of a type-II error, which occurs when a false null hypothesis is not rejected.

Values must be between zero and one. Historically, the value of 0.80 (Beta = 0.20) was used for power. Now, 0.90 (Beta = 0.10) is also commonly used.

A single value may be entered here or a range of values such as 0.8 to 0.95 by 0.05 may be entered.

**Alpha**

This option specifies one or more values for the probability of a type-I error (alpha). A type-I error occurs when you reject the null hypothesis when in fact it is true.

Values of alpha must be between zero and one. Historically, the value of 0.05 has been used for alpha. This means that about one test in twenty will falsely reject the null hypothesis. You should pick a value for alpha that represents the risk of a type-I error you are willing to take in your experimental situation.

You may enter a range of values such as 0.01 0.05 0.10 or 0.01 to 0.10 by 0.01.

**Sample Size**

**N (Sample Size)**

This option specifies the value(s) for \( N \), the sample size. Note that \( 3 < N \).

**Effect Size**

**\( \beta_m \) (Reg Coef of M)**

Enter one or more values for \( \beta_m \) which is the coefficient of M in the Poisson regression model

\[
\log(Y) = \beta_0 + \beta x(X) + \beta_m (M)
\]

Y is the count (response), X is the primary predictor variable, and M is the mediator variable.

Range

\( \beta_m \) can be any value other than zero.

**\( \rho_{X M} \) (Correlation of X and M)**

Enter one or more values for the correlation of X and M.

Range

\( 0 < \rho_{XM} < 1 \).
Data Types of X and M
Specify the data types (continuous or binary) of independent variables X and M.
X is the primary predictor variable.
M is the mediator variable.

Note
Here, 'Continuous' refers to data that are normally distributed.

σₓ (Standard Deviation of X)
Enter one or more values for the standard deviation of X, the primary predictor variable.
Range
0 < σₓ.

σₘ (Standard Deviation of M)
Enter one or more values for the standard deviation of X, the primary predictor variable.
Range
0 < σₘ.

Probability X = 1
Enter one or more values of the probability in the population that X = 1 when X is a binary variable that takes on the values 0 or 1. This value is used to compute the variance of X using the formula: σₓ² = P(X=1) × (1 - P(X=1)).
Range
Since this is a probability, it can be any value between 0 and 1.

Probability M = 1
Enter one or more values of the probability in the population that M = 1 when M is a binary variable that takes on the values 0 or 1. This value is used to compute the variance of M using the formula: σₘ² = P(M=1) × (1 - P(M=1)).
Range
Since this is a probability, it can be any value between 0 and 1.

E(Y) (Marginal Mean of Y)
Enter one or more values for the marginal mean of Y, where Y is the count variable. Y cannot be zero since the model uses log(Y).
Range
0 < E(Y).
Example 1 – Finding Sample Size

Researchers are studying the relationship between a count-type dependent variable (Y) and a continuous independent variable (X). They want to understand the impact of a continuous third variable (M) on the relationship between X and Y, so they decide to carry out a mediation analysis. They decide to determine the sample size based on the significance test of the mediator term in a Poisson regression. Using prior analyses, they decide to use $\beta_M = 0.3, 0.4, 0.5$, $\rho_{XM} = 0.4$, $\sigma_M = 1$, and $E(Y) = 0.5 \text{ to } 0.8$. They set the power at 0.9 and the two-sided significance level at 0.05.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the Tests of Mediation Effect in Poisson Regression procedure. You may then make the appropriate entries as listed below, or open Example 1 by going to the File menu and choosing Open Example Template.

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design Tab</td>
<td>N (Sample Size)</td>
</tr>
<tr>
<td>Solve For</td>
<td></td>
</tr>
<tr>
<td>Alternative Hypothesis</td>
<td>Two-Sided</td>
</tr>
<tr>
<td>Power</td>
<td>0.90</td>
</tr>
<tr>
<td>Alpha</td>
<td>0.05</td>
</tr>
<tr>
<td>Data Types of X and M</td>
<td>X = Continuous, M = Continuous</td>
</tr>
<tr>
<td>$\beta_M$ (Reg Coef of M)</td>
<td>0.3 0.4 0.5</td>
</tr>
<tr>
<td>$\rho_{XM}$ (Correlation of X and M)</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sigma_M$ (Standard Deviation of M)</td>
<td>1</td>
</tr>
<tr>
<td>$E(Y)$ (Marginal Mean of Y)</td>
<td>0.5 0.8</td>
</tr>
</tbody>
</table>

Annotated Output

Click the Calculate button to perform the calculations and generate the following output.

<table>
<thead>
<tr>
<th>Numeric Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Types: X is continuous and M is continuous</td>
</tr>
<tr>
<td>Two-sided alternative hypothesis</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Power</th>
<th>Sample Size</th>
<th>Reg Coef of M</th>
<th>Corr of X and M</th>
<th>Std Dev of M</th>
<th>E(Y)</th>
<th>Alpha</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9000</td>
<td>278</td>
<td>0.300</td>
<td>0.400</td>
<td>1.00</td>
<td>0.50</td>
<td>0.050</td>
</tr>
<tr>
<td>0.9004</td>
<td>174</td>
<td>0.300</td>
<td>0.400</td>
<td>1.00</td>
<td>0.80</td>
<td>0.050</td>
</tr>
<tr>
<td>0.9012</td>
<td>157</td>
<td>0.400</td>
<td>0.400</td>
<td>1.00</td>
<td>0.50</td>
<td>0.050</td>
</tr>
<tr>
<td>0.9008</td>
<td>98</td>
<td>0.400</td>
<td>0.400</td>
<td>1.00</td>
<td>0.80</td>
<td>0.050</td>
</tr>
<tr>
<td>0.9026</td>
<td>101</td>
<td>0.500</td>
<td>0.400</td>
<td>1.00</td>
<td>0.50</td>
<td>0.050</td>
</tr>
<tr>
<td>0.9021</td>
<td>63</td>
<td>0.500</td>
<td>0.400</td>
<td>1.00</td>
<td>0.80</td>
<td>0.050</td>
</tr>
</tbody>
</table>

References

Report Definitions
Hypotheses: H0: βₓ = 0 versus H1: βₓ ≠ 0. (Two-Sided).
X is the primary predictor. It is a continuous variable.
M is the mediator. It is a continuous variable.
Model: log(Y) = β₀ + βₓ(X) + βₓ(M).
Power is the probability of rejecting a false null hypothesis.
N is the number of observations on which the multiple regression is computed.
βₓ is the regression coefficient of the mediator in the model.
ρₓₓ is the correlation between X and M.
σₓ is the standard deviation of M.
E(Y) is the marginal mean of the outcome, Y.
Alpha is the probability of rejecting a true null hypothesis. It should be small.

Summary Statements
A sample size of 278 achieves 90% power to detect a mediation effect (as measured by the regression coefficient of M, βₓ) of at least 0.300 when the two-sided significance level (alpha) is 0.050. The correlation between X (the primary predictor) and M (mediator) is 0.400. The mediator, M, has a standard deviation of 1.00. The marginal mean of the outcome, E(Y), is 0.50.

This report shows the necessary sample sizes. The definitions of each of the columns is given in the Report Definitions section.

Plots Section

These plots show the relationship between sample size and the regression coefficient.
Example 2 – Validation using Vittinghoff (2009)

Vittinghoff et al. (2009) present an example on page 546 in which $\beta_X = \log 1.4 (0.3365)$, $\beta_M = \log 1.35 (0.3001)$, $\rho_{XM} = 0.5$, $\sigma_X = 1$, $P(M=1) = 0.25$, and $E(Y) = 0.5$. They set the power at 0.8 and the one-sided significance level at 0.025. The computed sample size is 1037.

Setup

This section presents the values of each of the parameters needed to run this example. First, from the PASS Home window, load the Tests of Mediation Effect in Poisson Regression procedure. You may then make the appropriate entries as listed below, or open Example 2 by going to the File menu and choosing Open Example Template.

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve For</td>
<td>N (Sample Size)</td>
</tr>
<tr>
<td>Alternative Hypothesis</td>
<td>One-Sided</td>
</tr>
<tr>
<td>Power</td>
<td>0.80</td>
</tr>
<tr>
<td>Alpha</td>
<td>0.025</td>
</tr>
<tr>
<td>Data Types of X and M</td>
<td>X = Continuous, M = Binary</td>
</tr>
<tr>
<td>$\beta_X$ (Reg Coef of X)</td>
<td>0.3365</td>
</tr>
<tr>
<td>$\beta_M$ (Reg Coef of M)</td>
<td>0.3001</td>
</tr>
<tr>
<td>$\rho_{XM}$ (Correlation of X and M)</td>
<td>0.5</td>
</tr>
<tr>
<td>$\sigma_X$ (Standard Deviation of X)</td>
<td>1</td>
</tr>
<tr>
<td>Probability M = 1</td>
<td>0.25</td>
</tr>
<tr>
<td>E(Y) (Marginal Mean of Y)</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Output

Click the Calculate button to perform the calculations and generate the following output.

Numéric Results

<table>
<thead>
<tr>
<th>Numeric Results</th>
<th>Data Types: X is continuous and M is binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reg Coef of X N</td>
<td>0.337</td>
</tr>
<tr>
<td>Reg Coef of M N</td>
<td>0.300</td>
</tr>
<tr>
<td>Corr of X and M N</td>
<td>0.500</td>
</tr>
<tr>
<td>Std Dev of X N</td>
<td>1.00</td>
</tr>
<tr>
<td>Prob M=1 N</td>
<td>0.250</td>
</tr>
<tr>
<td>Marg Mean of Y N</td>
<td>0.50</td>
</tr>
<tr>
<td>Alpha N</td>
<td>0.025</td>
</tr>
</tbody>
</table>

PASS matches the calculation of N = 1037.